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PRACTICAL MATHEMATICS.

SUMMARY OF SIX LECTURES DELIVERED TO WORKING MEN BY

PROFESSOR JOHN PERRY, D.Sc., F.R.S.,

AT

THE MUSEUM OF PRACTICAL GEOLOGY,

JERMYN STREET,

FEBRUARY AND MARCH, 1899,

WITH CERTAIN EXERCISES SUPPOSED TO BE WORKED AFTER EVERY LECTURE.

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PREFACE.

PRACTICAL MATHEMATICS having found a place in the "Science and Art Directory," Professor Perry was asked to summarise, for the benefit of teachers and students, six lectures on that subject which he had delivered at Jermyn Street to working men. He has compiled the following pages with great care, and they indicate the scope of the subject which it is the wish of the Department of Science and Art to encourage in the Syllabus issued in the "Directory." The subject of Practical Mathematics is intended as a preliminary study to the different branches of Mechanics for those who cannot afford time to go through a longer course of instruction in Mathematics.

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PRACTICAL MATHEMATICS.

LECTURE I.

1. Introductory remarks upon Education.* Great reforms needed. The most prominent Englishmen understand nothing of those sciences which are transforming all the conditions of civilization. Just as a board-school boy is allowed now to perform arithmetical work which was quite beyond the Alexandrian Philosophers, and no one dreams of philosophising over Euclids 7th, 8th, 9th, and 10th books; so the other books of Euclid must be superseded. For the whole of the 5th book of Euclid we need only half a page of easy algebra.

2. Training in Practical Mathematics begins with **Arithmetic**, the use of **decimals** being introduced at the very beginning.

3. When calculating from observed quantities, it is dishonest to use **more figures** than we are sure of.

Exercise.—For a certain purpose it is necessary to measure two distances in inches, to multiply the numbers and to extract the square root. Find the answer when the distances are 2·34 and 1·56 inches. Find the answers also when the distances are 2·33 and 1·55; 2·35 and 1·57; 2·35 and 1·55. *Ans.* 1·9106, 1·9004, 1·9208, 1·9085.

If the student will suppose the method of measurement of the distances to be such that an error of ·01 of an inch was possible, he will see that only three figures, say 1·91 ought to be given in the answer.

In a leading newspaper a few days ago I saw the indicated horse-power of a marine engine quoted as 3562·74 horse-power. Well, it is very probable that this measurement is in error at least 5 per cent. That is, the person who made the measurements and calculations is not sure whether the answer might not be 3700 or 3400, and yet he pretends that this last figure has a meaning. I am sorry to say that many misleading figures of this kind are published in the best books written on the steam engine.

* The introductory remarks occupied much of the time of the first lecture These it is unnecessary to reproduce here.

When I was at school the mean distance from the earth to the sun was stated as 95,142,357 miles. I wonder why furlongs and inches were not mentioned. The best knowledge we now have of this distance is that it is not greater than 93 nor less than 92½ millions of miles.

4. It will greatly prevent this sort of error if students will get into the way of writing $9{\cdot}3 \times 10^7$ or $9{\cdot}25 \times 10^7$ instead of 93,000,000 or 92,500,000. This is convenient in other ways. In the following table, note that we count how many places the decimal point must be moved to convert 5·204 into the number in question:—*

520,400,000,000	$5{\cdot}204 \times 10^{11}$
520,400	$5{\cdot}204 \times 10^{5}$
52·04	$5{\cdot}204 \times 10^{1}$
·5204	$5{\cdot}204 \times 10^{-1}$
·005204	$5{\cdot}204 \times 10^{-3}$

When I calculate I seldom trouble my head about the position of the decimal point in my answer until everything else is finished. There are many cleverly contrived rules about the position of the decimal point, but we forget them in practical work. Better never learn them.

5. Multiplication.—In multiplying 5623 × 1547, if we only want four figures in the answer, I show the ordinary method. Then I put 0 for the unnecessary figures; then I show how one actually works. Let a student study, understand, and practise for himself.

5623	5623	5623
1547	1547	7451
39361	39000	5623
22492	22500	2812
28115	28120	225
5623	5623	39
8698781	8699	8699

The multiplier is written backwards. In multiplying by the 4, say, we are supposed to multiply only on the 56; the figure above the 4 and all to the left of it; but we add 1 because 4 × 2 = 8, and we cannot reject 8, we call it 10, that is we carry 1. Students must practice and get very familiar with the method. Some men prefer not to write the multiplier backwards, their work being however what I have given.

* Sometimes the following shortened forms of the above numbers are found useful: 950^6, 9250^5, 52040^3, &c., the last being $0^2 5204$. Also 00000005204 would be written $0^7 5204$.

6. Division.—Divide 5623 by 1547. Here is the ordinary method: All figures to the right of the dotted line are unnecessary; hence the merit of the other method in which we do not add 0's, but cut off figures in the divisor.

1547) 5623 | 3634 77 &c.	1547) 5623 | 3635
4641	4641
9820	982
9282	928
5380	54
4641	46
7390	8
6188	8
A —— B	——
12020	
10829	
11910	
10829	
1081	

There is a further contraction of work, the above numbers 4641, 928, 46, 8, not being written, which ought to be taught to all boys; but grown up people do not take to it readily as a new method.

7. Use of Formulæ.—Mathematical symbols are merely a very easy form of shorthand; they usually instruct us to perform certain arithmetical operations. Thus +, −, × and ÷ are well known. Of course $a \times b$ means the same as $a.b$ or ab when letters are used to represent quantities, but only the first of these ought to be used with numbers, for very obvious reasons. Again $a \div b$ or $a : b$ or $\frac{a}{b}$ or a/b mean the same. The use of brackets ought to be familiar to students. Again $\sqrt{a}$ is the same as $a^{\frac{1}{2}}$; $\sqrt[3]{a}$ the same as $a^{\frac{1}{3}}$; $a^{\frac{2}{3}}$ the same as $\sqrt[3]{a^2}$, or by a_q^p we mean the q^{th} root of the p^{th} power of a or the p^{th} power of the q^{th} root of a. Again a^{-m} means $1 \div a^m$.

Sometimes in a formula we find a term like sin Θ or cos Θ or tan Θ, and this means that Θ being some given angle we are asked to look up the arithmetical value of the sine of this angle, or its cosine, or its tangent, in some suitable table, and use it in calculating with the formula.

Again $\sin^{-1}a$ means "the angle whose sine is a," and if we know a it is easy from tables to find $\sin^{-1}a$, $\cos^{-1}a$, $\tan^{-1}a$, &c. It is a great pity that this symbol should be liable to be taken for another.

Again if we use $\log_{10}x$ we know that we are expected to look up the value of the common logarithm of the number x in a table of common logarithms or logarithms to the base 10.

Again if we see $\log_e x$ we know that we are required to use the Napierian, or, as same people call it, the Natural, or, as others call it, the Hyperbolic Logarithm of the number x; a mathematician usually means $\log_e x$ when he writes $\log x$.

Some formulæ tell us to look up other things, but there are always sufficient instructions to enable the necessary arithmetical work to be carried out if a teacher will only give exercises to his pupils and say nothing of philosophical difficulties which exist only in his imagination. The pupils have usually enough common sense to follow the plain instructions of even complicated formulæ.

The accompanying tables will enable the following exercises to be worked. But some of the arithmetical work will be more easily done after the next section is read.

We often use other symbols in our work: $a > b$ means "a is greater than b"; $a < b$ means "a is less than b"; $y \propto x$ means "y varies as x, or y is proportional to x, that is, y is equal to x multiplied by some constant number."

8. Logarithms.—The use of logarithms enables us to compute much more rapidly than by ordinary arithmetic. Many kinds of computation seem almost hopelessly difficult except by the use of logarithms.

The symbol a^3 means $a \times a \times a$

Hence $2^3 = 8$; $2^5 = 32$

Many people say "a^3 means a multiplied on itself three times." Of course this is wrong. It is, however, right to say "a^3 means 1 multiplied by a three times." Thus a^0 means 1 multiplied by a no times.

Definition of a Logarithm.

If $a^n = N$ then $n = \log_a N$, and we read this as "n is the logarithm of N to the base a." Thus $2^3 = 8$, and hence 3 is the logarithm of 8 to the base 2.

We almost always use only logarithms to the base 10 in arithmetical work because we use the decimal system of writing numbers; but in many important calculations we need to use Napierian logarithms whose base is 2·71828, a number so important that the letter e is generally used to

denote it, just as the Greek letter π is used to denote 3·14159. It can be shown that if we multiply the common logarithm of any number by 2·30258 or divide by 0·43429 we get its Napierian logarithm.*

*NOTE.—If $e^x = N = 10^n$ then $x = \log_e N = n \log_e 10$ or $n = \log_{10} N = x \log_{10} e$ so that $x = n \div \log_{10} e$, and it will be found that the common logarithm of e or 2·71828 is 0·43429.

We know that $10^a \times 10^b = 10^{a+b}$

And $10^a \div 10^b = 10^{a-b}$

Also $10^a \div 10^a = 10^0 = 1$.

A student ought to illustrate this for himself by extracting a very high root of any number, and he will find that the higher the root the more nearly does the answer approach 1.

Also $1 \div 10^a = 10^{0-a} = 10^{-a}$.

We have then the rules:—

1. Add the logarithms of two numbers and we have the logarithm of their product.

2. Subtract the logarithms of two numbers and we have the logarithm of their quotient.

Again, the logarithm of a^2 is *twice* the logarithm of a.

" " " a^3 " *three* times " " " a.

" " " $a^{\frac{1}{2}}$ or $\sqrt{a}$ is *half* " " " a.

" " " $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$ " *one-third* the " " a.

In the same way if b is any number whatsoever the logarithm of a^b is b times the logarithm of a.

9. I have given each of you a copy of a set of tables of four figure logarithms and antilogarithms published by the Department of Science and Art; there is also a table of the sines, cosines, tangents and radian measure of angles between 0° and 90°, and the sixth page of the little pamphlet gives various useful numbers and formulæ.

In Lecture III. I hope to tell you how to calculate a table of logarithms.

10. Now to use the table headed "logarithms," notice that there are no decimal points anywhere. Given the number 5204 we look for 52 on the left and 0 above, this gives us 7160; our 4 causes us to look for the small number on the right hand columns, and we add it and so get 7163. The smallest amount of practice is surely enough to teach this.

Now I want you to understand that this means

$$\log 5{\cdot}204 = 0{\cdot}7163$$

You will see therefore that to find the logarithm of any number it is only necessary to study the following illustrative examples. Multiplying a number by 10 adds 1 to its logarithm.

Number.	Number as written in Page 4.	The logarithm is	The value of the logarithm is more compactly written.
520400	$5{\cdot}204 \times 10^{5}$	·7163 + 5	5·7163
5204	$5{\cdot}204 \times 10^{3}$	·7163 + 3	3·7163
5·204	$5{\cdot}204 \times 10^{0}$	·7163 + 0	0·7163
·5204	$5{\cdot}204 \times 10^{-1}$	·7163 − 1	$\bar{1}$·7163
·005204	$5{\cdot}204 \times 10^{-3}$	·7163 − 3	$\bar{3}$·7163

The student notices that the whole number part of a logarithm depends on the position of the decimal point in the number.

If you like in 5·7163 or $\bar{3}$·7163 you may call the 5 or $\bar{3}$ (the whole number part) by such names as *index* or *characteristic*, and **the decimal part**, which **is always positive**, you may call the mantissa; but in truth you will do much better if you scorn the use of unnecessary technical terms like these.

11. Again given the logarithm of a number to find the number.* Proceed backwards as from column 4 to column 1 of last table.

Thus to find the number whose logarithm is 3·7163. In the antilogarithm table (we might do it with but little increase of trouble by means of the logarithm table itself), find what corresponds to ·7163; it is evidently 5200 + 4 or 5204. This means that

$$\log 5{\cdot}204 = {\cdot}7163$$

and hence $\log 5204 = 3{\cdot}7163$

12. A little practice in multiplication and division will make you perfectly familiar with and give you a thorough understanding of this subject. Without such practice it is as useless to read minute instructions as it is to try to learn to swim or ride a bicycle by reading instructions.

If we are asked to multiply or divide a logarithm itself it is necessary to remember that the whole number part is sometimes negative.

* Why not say;—*delogarize* 3·7163, as companion to *logarize* 5204

Exercise. Find the values of

$$\sqrt{8574} \quad \sqrt{8{\cdot}754} \quad \sqrt{0{\cdot}8754} \quad \sqrt{0{\cdot}0008754}$$

The logarithms of these numbers have to be divided by 2, as the logarithm of a square root is just half the logarithm of the number.

Number N.	log N.	½ log N.	Answer.
8574	3·9332	1·9666	92·60
8·574	0·9332	0.4666	2·928
0·8574	$\bar{1}$·9332	$\bar{1}$·9666	·9260
0·0008574	$\bar{4}$·933	$\bar{2}$·4666	·02928

13. Thus to divide $\bar{1}$·9332 by 2, I remember that it is really $-1 + {\cdot}9332$; this is the same as $-2 + 1{\cdot}9332$, and the half of this is $-1 + {\cdot}9666$ or $\bar{1}$·9666.

To divide $\bar{1}$·9332 by 3, I call it $-3 + 2{\cdot}9332$, so that the answer is $-1 + 0{\cdot}9777$, or $\bar{1}$·9777.

To divide or multiply $\bar{3}$·9332 by such a number as 4·56, it is necessary to convert it all into the negative form. It is $-3 + 0{\cdot}9332$ and this is really $-2{\cdot}0668$. Dividing this by 4·56 we get $-0{\cdot}4532$, and this is $-1 + 0{\cdot}5468$ or $\bar{1}$·5468.

Thus $(\cdot00857)^{\frac{1}{4{\cdot}56}}$ is 0·3522.

14. The Slide Rule.—To multiply 5623×1547 we say

log. 5623=3·7499
log. 1547=3·1895
adding 6·9394=logarithm of $8{\cdot}698 \times 10^6$

The essential part of the work is the adding of ·7499 and ·1895, and we can add numbers mechanically in many ways.

It would be laborious to put 7499 beans in a bag and then put in 1895 and count the whole. We might put 749·9 ounces and 189·5 ounces in a scale pan and weigh the lot, thus getting at the sum. A good plan is to measure off the distance 7·499 inches or centimetres on a scale as in Fig. 1, measure off the distance 1·895 inches or centimetres on another scale; and place the two distances so that their sum may be measured.

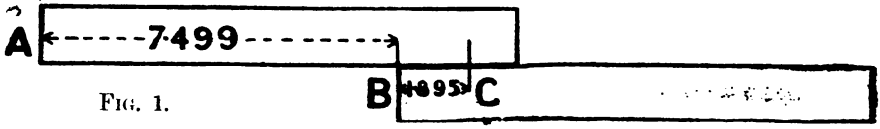


Fig. 1.

Fig. 1, the distance AB added to the distance BC is the distance AC.

Now the slide rule does this very thing, only the slide rule has not the numbers of Fig. 1 written upon it. It has two sliding scales, and in the above position of things the marks upon the scale would be those shown in Fig. 2;

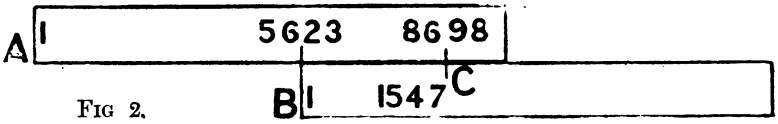


FIG 2.

that is, although the distance from A to B represents 7·499 inches or ·7499 to some scale of measurement or other; it is not ·7499 that is written on the upper scale but the number 5623 or 5·623 of which ·7499 is the logarithm and it is not the logarithm ·9394 which we read off opposite C on the upper scale, although this really is the distance from A to C; it is the number 8698 or 8·698 which we see there.

Perhaps if I make a slide rule before you it will be easier to comprehend. Here are two black scales, A and B, which slide past one another. Making their ends agree, I am going to put the same marks on both. I have a sort of tape line here, by taking a certain distance (you need not trouble yourselves about its actual length in inches) which I shall call a unit distance. I have divided it up into 10 and 100 equal parts so as to be able to set off any distance less than unity on my scales. To know where to put the number 4·5 on my scale I have made its distance from the point 1 to be ·6532. I have put the number 5 at a distance from 1 equal to ·6990.

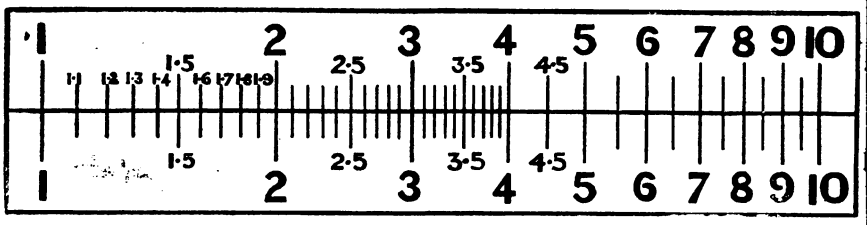


FIG 3.

Number on Scale.	At a distance from the mark 1 of	Because
4.5	·6532	log. 4·5 = ·6532
5.0	·6990	log. 5.0 = ·6990
10·0	1·0000	log. 10 = 1·0000

In fact my distances from the mark 1 are proportional to the logarithms of the numbers placed at those distances.

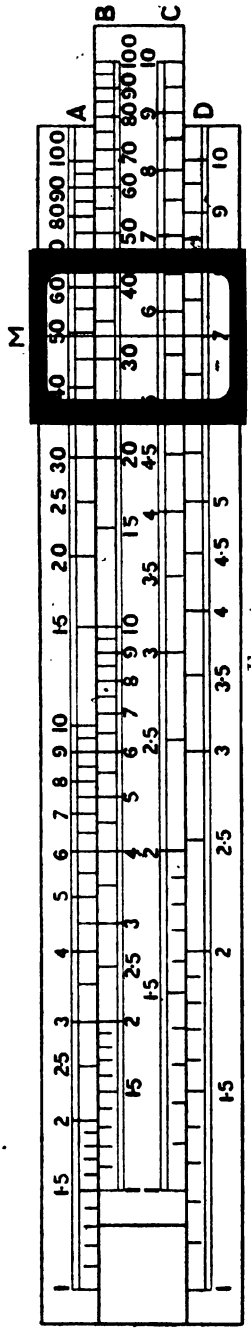


FIG. 4.

A slide rule has several scales. If you have a slide rule, Fig. 3, look at two marked A and B. Slide them so that they agree exactly. You will find that the distance from the mark 1 to the mark 2 or 3 or 4 or 12 is not 2 inches or 3 inches or 4 inches or 12 inches, but really ·3310 inch, ·4771 inch, ·6021 inch, or 1·0792 inch, because these four numbers are the *logarithms* of 2, 3, 4, 12. So if you place 1 of B against 3 of A, and look for the number on A, which is opposite 4 of B, you will see 12. You have **multiplied** 3×4 because you have merely added ·4771 and ·6021 to get 1·0792. Think it out for yourself; practise multiplying simple numbers; ask nobody to help you, and you will rapidly get familiar with and fond of the slide rule.

The lecturer manufactured a slide rule in two minutes, sufficiently good to illustrate multiplication and division. Every student ought to manufacture a slide rule, using two strips of paper.

To **divide** 12 by 4. Opposite 12 of A, place 4 of B, and the answer is on A, opposite 1 of B. Again practise division.

Multiply $4 \times 3 \times 6 \times 7$, and so learn the value of the sliding marker (called a cursor). When you have multiplied 4×3, put the marker at the answer. You do not want to know this answer. Now put 1 of B at the marker, and remove the marker to 6 of B and so on. If you practise by yourself, you will need no telling. Again, if your answer is beyond the end of scale A, slide B back until the mark 10 or 100 occupies the place that 1 occupied; this is just as if you lengthened scale A. But instructions are of no use. Find all this out for yourself.

How do you find the **reciprocal** of a number: divide 1 or 10 by the number; it is easy enough to do. Notice that when 1 of B is opposite 3 of A, then every number of B is opposite 3 times it of A. Thus one position of B enables us to multiply all the numbers in a table by the same number.

Test for yourself your accuracy in multiplying and dividing. Note how curiously the scales are divided, and accustom yourself to reading off numbers quickly.

In an actual slide rule, Fig. 4, we have not only the scales A and B which slide alongside one another, but another pair C and D which are alongside one another. These are prepared in the same way as A and B, only that

if the distance from 1 to 2 on scale A or B is ·3010 inch

then „ „ 1 to 2 „ „ C or D is ·6020 „

You will see therefore that a number on D is just underneath its square on A, and you will see that the sliding marker M enables you to find **squares** or **square roots.**

Also it is easy to multiply or divide any number by the square or square root of any other number, and this again ought to be practised. So again any number may be multiplied by its own square and so we get its **cube** and the student will see that we can reverse this process and so extract a **cube root.**

There are some very important calculations which cannot be easily done with the slide rule, the most important being the finding of a^b where a and b are any numbers whatever. Read no book of elaborate instructions, you can find out everything for yourself by using the rule. Here on the lecture table are numerous contrivances called by all sorts of names which are really all slide rules. Notice how the Fuller rule instead of being straight, is spiral, so that we get what is equivalent to an exceedingly long rule in a small compass. Also these tables of Professor Everett are really a very long rule in the shape of two sheets of card board.

Exercises.—(1.) Calculate ab and a/b when $a = 1323$ and $b = 24{\cdot}32$. *Ans.*, 32175, 54·4.

Again, when $a = 17{\cdot}56$ and $b = 143{\cdot}5$. *Ans.*, 2520, 0·1224.

Again, when $a = 0{\cdot}5642$ and $b = 0{\cdot}2471$. *Ans.*, 0·1394, 2·283

(2.) Show that the following statements of the standard taken as the pressure of one atmosphere agree:—

14·70 lb. per square inch; 2116 lb. per square foot.
29·92 inches of Mercury at 0° C., 760 mm. of Mercury at 0° C.
1033 kilos per square metre; 33·9 feet of pure water at 0° C., 33·04 feet of sea-water at 0° C.

(3.) Calculate a^b. That is, the number a raised to the power indicated by b.

Find the logarithm of a, multiply it by b, and this is the logarithm of the answer:—

Let $a = 20{\cdot}52$ and $b = 2$.	*Ans.*, 421·1.
„ $a = 1{\cdot}564$ „ $b = 1\frac{1}{2}$.	„ 1·956.
„ $a = 0{\cdot}5728$ „ $b = 3$.	„ 0·1879.
„ $a = 60{\cdot}71$ „ $b = \frac{1}{3}$.	„ 3·930.

Note here that to multiply by $\frac{1}{3}$ means that we are to divide by 3.

$a = 0{\cdot}2415$ and $b = \frac{1}{3}$.	*Ans.*, 0·6227.
$a = 1{\cdot}671$ „ $b = 2$.	„ 0·3581.
$a = {\cdot}5014$ „ $b = 3\frac{1}{2}$.	„ 0·08921.

(4.) Find $4{\cdot}326 \times 0{\cdot}003457$ to four significant figures, leaving out all unnecessary figures in the work.

Find $0{\cdot}01584 \div 2{\cdot}104$ to four significant figures.

Also do these using logarithms. Find $\log_e 7$.

Calculate $5^{2\cdot43}$, $3^{-0\cdot246}$, $\cdot042^{0\cdot476}$, $\sqrt[5]{246{\cdot}3}$, $30{\cdot}01^{\frac{2}{3}} \times 0{\cdot}02641^{\frac{1}{4}}$.

Ans. 0·01495, 0·007529, 1·94591, 49·95, 0·7632, 0·2211, 3·008, 5·872.

(5.) If $m = (a^3 + 2a^2b + S^{-\cdot345})^{\frac{2}{3}} \div (a^2b^2)^5$, find m, if $a = 0{\cdot}504$, $b = 0{\cdot}309$, $S = 1{\cdot}567$. *Ans.* $1{\cdot}453 \times 10^8$.

(6.) When a is very small compared with 1 we may write $(1 + a)^n = 1 + na$, very nearly. Employ this to show that the following relations are nearly correct:—

$(1{\cdot}001)^3 = 1{\cdot}003$; $(1{\cdot}01)^{\frac{1}{3}} = 1{\cdot}0033$; $(0{\cdot}99)^2 = (1 - {\cdot}01)^2 = 1 - {\cdot}02 = {\cdot}98$; $\frac{1}{{\cdot}99} = \frac{1}{1 - {\cdot}01} = (1 - {\cdot}01)^{-1} = 1 + {\cdot}01 = 1{\cdot}01$.

$$\frac{1}{\sqrt[3]{1{\cdot}01}} = (1 + {\cdot}01)^{-\frac{1}{3}} = 1 - {\cdot}0033 = {\cdot}9967$$

$$\sqrt{99} = \sqrt{100\,(1 - {\cdot}01)} = 10\,(1 - {\cdot}01)^{\frac{1}{2}} = 10\,(1 - {\cdot}005) = 9{\cdot}95$$

Find what the exact percentage errors are in these answers. For example, find the cube of 1·001 by multiplication, and observe how slightly it differs from 1·003.

(7.) How much error is there in the assumptions $\frac{1+\alpha}{1+\beta} = 1 + \alpha - \beta$; $(1 + \alpha)(1 - \beta) = 1 + \alpha - \beta$ when $\alpha = \cdot 01$, $\beta = \cdot 01$, $\alpha = -\cdot 003$, $\beta = -\cdot 005$?

Ans. No error; ·01 per cent, ·004 per cent., ·0015 per cent.

(8.) (*a*) Work out the values of $M = (sr^{-1} - r^{-s})/(s-1)$ when $s = 0{\cdot}8$, and r has the values 1·333, 1·5, 2, 3, 5, 8, 12, 20.

(*b*) Work out the values of M in Exercise 8 if s is 1·2.

(*c*) Work out the values of $M = (1 + \log_e r)/r$ when r has the above values.

Exhibit the three sets of values in a table.

(9.) Instead of calculating $a + \sqrt{a^2 + b^2}$ we often substitute, as giving nearly the same answer, $1{\cdot}84a + 0{\cdot}84b$. Take $a = 1$, and taking various values of b, show the two sets of answers in a table. Within what limiting values of b may we assume the error to be less than 3 per cent. ?

(10.) The correct formula to use in a certain practical investigation is

$$y = 0{\cdot}75 + 2{\cdot}59x^2 + \sqrt{0{\cdot}5x^{-1}\sin 3x^2 + x\log_e x + 0{\cdot}3x^2} \quad (1).$$

It was known that it would not be used for values of x less than ·5 nor greater than 1.

It was quite impossible to work with it mathematically, so the following, found by means of squared paper, was used to replace it:

$$y = 0{\cdot}11 + 3{\cdot}833x \quad - \quad - \quad - \quad (2).$$

For various values of x compare the two formulæ.

Use your logarithms or slide rule or other method in evaluating any formulæ in any pocket book concerning your business.

EXAMINATION TABLES.

USEFUL CONSTANTS.

1 Inch = 25·4 millimetres.

1 Gallon = ·1605 cubic foot = 10 lb. of water at 62° F.

1 Knot = 6080 feet per hour.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 Cubic foot of water weighs 62·3 lb.

1 Cubic foot of air at 0° C and 1 atmosphere, weighs ·0807 lb.

1 Cubic foot of Hydrogen at 0° C and 1 atmosphere, weighs ·00557 lb.

1 Foot-pound = $1{\cdot}3562 \times 10^7$ ergs.

1 Horse-power-hour = 33000 × 60 foot-pounds.

1 Electrical unit = 1000 watt-hours.

Joule's Equivalent to suit Regnault's H, is $\begin{cases} 774 \text{ ft.-lb.} = 1 \text{ Fah. unit.} \\ 1393 \text{ ft.-lb.} = 1 \text{ Cent. ,,} \end{cases}$

1 Horse-power = 33000 foot pounds per minute = 746 watts.

Volts × amperes = watts.

1 Atmosphere = 14·7 lb. per square inch = 2116 lb. per square foot = 760 m.m. of mercury = 10^6 dynes per sq. cm. nearly.

A column of water 2·3 feet high corresponds to a pressure of 1 lb. per square inch.

Absolute temp., $t = \theta^\circ \text{ C.} + 273^\circ$.

Regnault's $H = 606{\cdot}5 + {\cdot}305\ \Theta^\circ \text{ C.} = 1082 + {\cdot}305\ \Theta^\circ \text{ F.}$

$p\, u^{1{\cdot}0646} = 479$

$$\log_{10} p = 6{\cdot}1007 - \frac{B}{t} - \frac{C}{t^2},$$

where $\log_{10} B = 3{\cdot}1812$, $\log_{10} C = 5{\cdot}0871$,

p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

$\pi = 3{\cdot}1416$.

One radian = 57·3 degrees.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithms is $e = 2{\cdot}7183$.

The value of g at London = 32·182 feet per sec. per sec.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	5	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1 1 2	3 4 4	5 6 7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1 2 2	3 4 5	5 6 7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1 2 2	3 4 5	5 6 7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1 2 2	3 4 5	6 6 7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1 2 2	3 4 5	6 6 7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1 2 2	3 4 5	6 7 7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1 2 3	3 4 5	6 7 8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1 2 3	3 4 5	6 7 8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1 2 3	4 4 5	6 7 8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1 2 3	4 5 5	6 7 8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1 2 3	4 5 6	6 7 8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1 2 3	4 5 6	7 8 9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1 2 3	4 5 6	7 8 9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1 2 3	4 5 6	7 8 9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1 2 3	4 5 7	7 8 9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1 2 3	4 5 6	7 8 9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1 2 3	4 5 6	7 9 10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1 2 3	4 5 7	8 9 10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1 2 3	4 6 7	8 9 10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1 2 3	5 6 7	8 9 10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1 2 4	5 6 7	8 9 11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1 2 4	5 6 7	8 10 11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1 2 4	5 6 7	9 10 11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1 3 4	5 6 8	9 10 11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1 3 4	5 6 8	9 10 12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 10 12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1 3 4	5 7 8	9 11 12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1 3 4	5 7 8	10 11 12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1 3 4	6 7 8	10 11 13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1 3 4	6 7 9	10 11 13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1 3 4	6 7 9	10 12 13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2 3 5	6 8 9	11 12 14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2 3 5	6 8 9	11 12 14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2 3 5	6 8 9	11 13 14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2 3 5	6 8 10	11 13 15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2 3 5	7 8 10	12 13 15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2 3 5	7 8 10	12 13 15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2 3 5	7 9 10	12 14 16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2 4 5	7 9 11	12 14 16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2 4 5	7 9 11	13 14 16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7 9 11	13 15 17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2 4 6	8 9 11	13 15 17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2 4 6	8 10 12	14 15 17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2 4 6	8 10 12	14 16 18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2 4 6	8 10 12	14 16 18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2 4 6	8 10 12	15 17 19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2 4 6	8 11 13	15 17 19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2 4 7	9 11 13	15 17 20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2 4 7	9 11 13	16 18 20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2 5 7	9 11 14	16 18 20

Angle.	Radians.	Sine.	Tangent.	Co-tangent.	Cosine		
0°	0	0	0	∞	1	1·5708	90°
1	·0175	·0175	·0175	57·2900	·9998	1·5533	89
2	·0349	·0349	·0349	28·6363	·9994	1·5359	88
3	·0524	·0523	·0524	19·0811	·9986	1·5184	87
4	·0698	·0698	·0699	14·3006	·9976	1·5010	86
5	·0873	·0872	·0875	11·4301	·9962	1·4835	85
6	·1047	·1045	·1051	9·5144	·9945	1·4661	84
7	·1222	·1219	·1228	8·1443	·9925	1·4486	83
8	·1396	·1392	·1405	7·1154	·9903	1·4312	82
9	·1571	·1564	·1584	6·3138	·9877	1·4137	81
10	·1745	·1736	·1763	5·6713	·9848	1·3963	80
11	·1920	·1908	1944	5·1446	·9816	1·3788	79
12	·2094	·2079	·2126	4·7046	·9781	1·3614	78
13	·2269	·2250	·2309	4·3315	·9744	1·3439	77
14	·2443	·2419	·2493	4·0108	·9703	1·3265	76
15	·2618	·2588	2679	3·7321	·9659	1·3090	75
16	·2793	·2756	·2867	3·4874	·9613	1·2915	74
17	·2967	2924	·3057	3·2709	·9563	1·2741	73
18	·3142	·3090	·3249	3·0777	·9511	1·2566	72
19	·3316	·3256	·3443	2·9042	·9455	1·2392	71
20	·3491	·3420	·3640	2·7475	·9397	1·2217	70
21	·3665	·3584	·3839	2·6051	·9336	1·2043	69
22	·3840	·3746	·4040	2·4751	·9272	1·1868	68
23	·4014	·3907	·4245	2·3559	·9205	1·1694	67
24	·4189	·4067	·4452	2·2460	·9135	1·1519	66
25	·4363	·4226	·4663	2·1445	·9063	1·1345	65
26	·4538	·4384	·4877	2·0503	·8988	1·1170	64
27	·4712	·4540	·5095	1·9626	·8910	1·0996	63
28	·4887	·4695	·5317	1·8807	·8830	1·0821	62
29	·5061	·4848	·5543	1·8040	·8746	1·0647	61
30	·5236	·5000	·5774	1·7321	·8660	1·0472	60
31	·5411	·5150	·6009	1·6643	·8572	1·0297	59
32	·5585	·5299	·6249	1·6003	·8480	1·0123	58
33	·5760	·5446	·6494	1·5399	·8387	·9948	57
34	·5934	·5592	·6745	1·4826	·8290	9774	56
35	·6109	·5736	·7002	1·4281	·8192	·9599	55
36	·6283	·5878	·7265	1·3764	·8090	·9425	54
37	·6458	·6018	·7536	1·3270	·7986	·9250	53
38	·6632	·6157	·7813	1·2799	·7880	·9076	52
39	·6807	·6293	·8098	1·2349	·7771	·8901	51
40	·6981	·6428	·8391	1·1918	·7660	·8727	50
41	·7156	·6561	·8693	1·1504	·7547	·8552	49
42	·7330	·6691	·9004	1·1106	·7431	·8378	48
43	·7505	·6820	·9325	1·0724	·7314	·8203	47
44	·7679	·6947	·9657	1·0355	·7193	·8029	46
45	·7854	·7071	1·0000	1·0000	·7071	·7854	45
		Cosine	Co-tangent	Tangent	Sine	Radians	Angle

LECTURE II.—ALGEBRA.

15. Just as it is quite easy for a beginner to learn to calculate from a formula, so it is usually easy for him to do many other useful things **with formulæ.**

Elementary Algebra is made difficult by the mere statement of rules. Why should any fuss be made over addition, subtraction and multiplication? Why, anybody who has used a formula with brackets knows these things already.

Tell a boy about ghosts and the simplest things become complex and mysterious. Tell a boy that he is sure to find difficulty in simple Algebra, and of course he finds great difficulty with a problem that would be quite easy if you told him that it was easy. I would give a boy **simple equation** work at once, and especially **problems** which are solved by simple equations, for there is not much that makes a boy think for himself so quickly as such work. I know that the average boy will learn quite rapidly and understand well simple equations in x, and also in x and y, and the problems leading to them.

Now is the time to show him how algebraic expressions are **simplified**, for he sees the usefulness of such a process; and especially of the importance of recognising the **factors** of such expressions as $x^2 - a^2$, $x^2 + 11x + 30$, $x^2 - 5x - 66$.

But where the greatest error is made in teaching is in not introducing to a student, quite early in all this work, as a sort of relief work, the **plotting of functions** by means of squared paper. For if he takes any function of x and calls it y, and for any value of x calculates y and plots the corresponding values of x and y on the squared paper, he gets a curve. If he now notes the value of x which makes the function 0, he gets in the simplest fashion the very best knowledge of the solution of an equation, of the root, of the roots of an equation, and he can and will philosophise on the subject without any prompting from his teacher.

I return to this matter later. But just here it is well to point out that when we have a formula of several algebraic quantities, it is often easy to express any one of the quantities in terms of the others so to make many useful calculations. In fact, then, I would ask teachers to mix together as one simple kind of algebraic work, easy to give, even to beginners, many parts of Algebra which are usually taken up very much later, so much later that the average student never reaches them in his study.

It will be seen then that I include in such work all sorts of work which goes under the name of "Rules in Arithmetic."

The following are only a few of the many hundreds of examples that may well be put before students.

16. I want a student to practise using all sorts of formulæ so that he shall cease to be afraid when he sees one. Of course, there may be some bit of shorthand, some symbol, which has not been explained yet to him, but he ought to know that there is nothing magical or uncanny about it. I have taken up some formulæ at random. I might call any one of them *a rule* and so create difficulty, but indeed there is only one way with them all.

The average man who has worked through many rules in complex arithmetic, and algebra, and engineering, very quickly forgets them all, except the one or two that he constantly needs. It is only a teacher who remembers hundreds of rules. But if at the beginning a man knows that his rules are **all one rule;** all his separate rules are mere examples of one general principle; he never can forget it, for every common-sense calculation that he makes only fixes the general principle more firmly in his mind.

Have you not noticed that a great man has only a few simple principles, on which to regulate all his actions? A great engineer keeps in his head just a few simple methods of calculation. But note that through constant practice these simple principles or methods are always ready for use in his mind. It may be that an expert may be quicker or neater in working some one kind of problem, but however clumsy or tedious may be the great man's method of working the problem, he gets the right answer, and he has no misgiving as he writes it out.

My one simple rule is to treat all numerical calculation as work with some formula, and all rules ought to be compactly stated as formulæ.

Of course, it is another thing to work out such formulæ, to prove them correct, and yet I say that even here, my general principle introduces enormous ease, for in most cases, to understand and feel unafraid of a formula is almost to see the proof of it.

17. Notice that if we have any equation, say $M = N$, we can say that $\log M = \log N$ or $M^n = N^n$ where n is any number. Again, if $x^n = y^m$ we may say that $n \log x = m \log y$.

Exercises.—(1.) If $pu^{1\cdot0646} = 479$. When $u = 4$ find p.

Here $\log p + 1\cdot0646 \log 4 = \log 479$, and we find $p = 109\cdot4$.

Again, when $p = 203$ find u. *Ans.*, 2·24.

(3.) Suppose $pv^{1\cdot13} = a$.

(α) If $p = 100$ and $v = 1$ find a. *Ans.*, 100.
If v becomes 1·5, using the same value of a find p. *Ans.*, 63·24.

(β) If v becomes 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4. In each case find the corresponding value of p.
Repeat the above work when $pv^{0\cdot9} = a$, taking $p = 100$ and $v = 1$ to start with.

(γ) Repeat again when $pv = a$.
Show the three sets of answers for p in a table.

(4.) If $p_1 v_1^{1\cdot13} = p_2 v_2^{1\cdot13}$ and if v_2/v_1 be called r. If $p_2 = 6$ find r for the following values of p_1:

p_1	250	200	150	100	50
Answers, r	27·13	22·27	17·26	12·06	6·53

(5.) Compare the formula in Exercise (1) with this—

$$\frac{1}{u} = \cdot0171 + \cdot0021p$$

by calculating u from both formulæ, for the following values of p, 80, 120, 140, 180, 220, and writing the two sets of answers in a table.

18. Proportion.—The Ratio of a to b or $a : b$ is $a \div b$. Thus the ratio of 2 to 3 is 0·6667.

When we say that y varies as, or is proportional to x, we write the statement in the form $y \propto x$ and we know that this is the same statement as $y = ax$ where a is some constant number.

Simple Proportion.—If $y \propto x$ and if $y = 4$ when $x = 3$ then as $y = ax$, $4 = a \times 3$ or, $a = \frac{4}{3}$, and hence $y = \frac{4}{3}x$ is the true law connecting y and x. If any value of x be now given, y may be found.

Compound Proportion.—If 10 labourers dig 15 yards of trench in 5 days of 12 hours; how many labourers will dig 356 yards in 2 days of 9 hours. If we use l, y, d, and h, we see that the assumption made is

$$l \propto \frac{y}{dh} \text{ or } l = a\frac{y}{dh}.$$

Hence $10 = a\dfrac{150}{5 \times 12}$ so that $a = \dfrac{5 \times 12 \times 10}{150} = 4$.

So that we have the formula $l = 4\,\frac{y}{dh}$ for working any exercise. Thus the answer wanted is

$$l = 4\,\frac{356}{2 \times 9} = 79{\cdot}11 \text{ labourers.}$$

Now notice that all the following exercises are worked in the same way.

Ex. 1. In steam vessels of the same character, I being horse-power, D displacement, v speed, $I \propto D^{\frac{1}{3}} v^3$.

(α) If $I = 655$ when $D = 1720$ tons and $v = 10$ knots, find the exact rule. *Ans.*, $I = D^{\frac{1}{3}} v^3 \div 219$.

(β) If D is 1500, $v = 13$, find I. *Ans.*, 1314.

(γ) If $I = 800$, $D = 1300$, find v. *Ans.*, 11·37 knots.

(1.) The weights W of similar objects are as the cubes of their like dimensions l, and as the densities w of their materials. Two similar objects are in the linear proportions of 1 to 4·37, and their densities are as 2 to 1·74. The weight of the first is 20 lbs. What is the weight of the second?

Ans., W = 1452 lbs.

(2.) If d is the calibre of a gun, it is usually assumed that the weight is proportional to d^3, and that the thickness of armour which its projectile will pierce is proportional to d. If an 8-inch gun weighs 14 tons, and can pierce 11 inches of armour, what thickness will be pierced by a 10-inch gun, and what is the weight of the gun?

Ans., 13·75 inches; 27·34 tons.

(3.) Two models of terrestrial globes are 1·22 feet and 3·14 feet diameter respectively. If the area of a country is 15 square inches on one, what is it on the other?

Ans., 99·32 square inches.

(4.) If $Q \propto h^{\frac{1}{2}}$ and if $Q = {\cdot}466$ when $h = 0{\cdot}5$, find the true law. *Ans.*, $Q = 2{\cdot}635\, h^{\frac{1}{2}}$.

(α) Find Q when $h = 1{\cdot}3$. *Ans.*, $Q = 5{\cdot}077$.

(β) Find h when $Q = 2{\cdot}46$. *Ans.*, $h = 0{\cdot}773$.

(5.) If we know that $y = a + bx$ where a and b are constant numbers; if $y = 12$ when $x = 1$, and if $y = 15$ when $x = 5$, find a and b.

Here we have $\begin{matrix} 12 = a + 1b \\ 15 = a + 5b \end{matrix}$. Subtracting we have

$3 = 4b$ or $b = \frac{3}{4}$, $12 = a + \frac{3}{4}$ or $a = 11\frac{1}{4}$.

Hence the law is really

$$y = 11\tfrac{1}{4} + \tfrac{3}{4}\, x.$$

(2.) Find y if $x = 4$. *Ans.*, $14\frac{1}{4}$.

(3.) Find x when $y = 20$. *Ans.*, $11\frac{2}{3}$.

(6.) If $y = ax + bz^{\frac{1}{3}}x^2$,

If $y = 49{\cdot}5$ when $x = 1$ and $z = 8$,

and $y = 356$ when $x = 1{\cdot}5$ and $z = 20$,

find a and b, and find the value of y when x is 2 and z is 17.

Ans., $y = -57{\cdot}1\,x + 26{\cdot}65\,z^{\frac{1}{3}}x^2$; 590·7.

19. Those parts of arithmetic called "Equation of Payments," "Barter," "Profit and Loss," "Fellowship," "Alligation" of many kinds, "Position," "Double Position," "Conjoined Proportion," and many others, are, when we strip them of their technical terms and artificial complexities, the simplest of algebraic exercises, and they ought to be treated as such.

20. Arithmetical **progression.** It is easy to show that

$$l = f + (n - 1)\,d, \text{ and } s = \tfrac{1}{2}\,n\,(f + l),$$

if there are n terms with common difference d (that is, any term minus the preceding term), if their sum is s, and if f is the first and l the last term.

It is, therefore, very easy to find any two of these when the other three are given.

(*a.*) Find the 15th term and the sum of 15 terms of ·25, ·50, ·75, &c.

(*b.*) What is the first term when the 59th term is 70 and the 66th term is 84.

(*c.*) Insert 4 arithmetic means between 3 and 18.

Geometrical Progression. If f, l, s, n are as in the last case; if r is the common ratio (that is, any term divided by the preceding term), it is easy to prove that

$$l = fr^{n-1},\ s = f\frac{r^n - 1}{r - 1}$$

(*d.*) Find the 10th term and the sum of 10 terms of 2, 6, 18, &c.

(*e.*) Insert 3 geometric means between 3 and 243.

(*f.*) The 6th term of a geometric series is 20·34, r is 0·26, find the first term and the sum of the 6 terms.

(*g.*) The sum of a geometric series of 5 terms is 534; r is 1·65; find the first and last terms.

21. Compound Interest.—It is very easy to prove that if a sum of money P (the principal) is lent at compound interest at r per cent. per annum, it amounts in n years to

$$\mathrm{A} = \mathrm{P}\left(1 + \frac{r}{100}\right)^n$$

and the increase $\mathrm{I} = \mathrm{A} - \mathrm{P}$ may be called the *interest.*

(1.) If P = 255·75, $r = 3\frac{1}{2}$, $n = 17$, find A.

Ans., 459.

(2.) If A = 930, $r = 4\frac{1}{2}$, $n = 10\frac{1}{2}$, find P.

Ans. 585·76.

(3.) If P = 320, I = 456, $r = 5$, find n.

Ans. 18·156.

(4.) If P = 250, A = 420, $n = 16$, find r.

Ans., 3·3.

(5.) P = one farthing, $n = 1900$, $r = 5$, find A.

Ans., 2×10^{37} pounds nearly.

(6.) Find n if A = 2 P.

$$Ans.,\ n = \log 2 \div \log\left(1 + \frac{r}{100}\right).$$

In Lecture III. we find by squared paper that $n = 70 \div r$ is a good approximation, much used by practical men.

(7.) If interest is added on m times a year

$$A = P\left(1 + \frac{r}{100m}\right)^{nm}.$$

(8.) If interest is added every moment $A = Pe^{nr/100}$.

The population of England and Wales doubles itself every fifty years; what is the rate per cent. per annum of increase?

A = 2 P if $n = 50$, and hence $r = 1{\cdot}386$.

22. Present Value and **Discount**. Use the formula for interest if the sum A is due at the end of n years and P is its present value. Call A − P the discount.

23. It is easy to prove that an **annuity** of £a per annum, payable for n years, has a present value £P, or will amount to £A if the use of money is worth r per cent. per annum, if

$$\frac{r\mathrm{A}}{100a} = \left(1 + \frac{r}{100}\right)^{n} - 1, \text{ or if } \frac{r\mathrm{P}}{100a} = 1 - \left(1 + \frac{r}{100}\right)^{-n}.$$

One of these formulæ is enough if it is remembered that the connection between A and P is what was given in the rule for compound interest.

24. Students are asked to take up numerical exercises on all these. Such exercises are to be found in many books. But indeed almost any pocket book formula serves for the creation of good exercises. The Mensuration Exercises of Art. 130 are supposed to be illustrations of the work of this lecture.

Exercise. Find $\frac{x}{y}$ from this equation $x^4 = 0{\cdot}8\,\frac{x^5 + y^5}{x}$.

Multiply by x and divide by y^5 all across and find $\frac{x}{y} = 1{\cdot}319$.

LECTURE III.

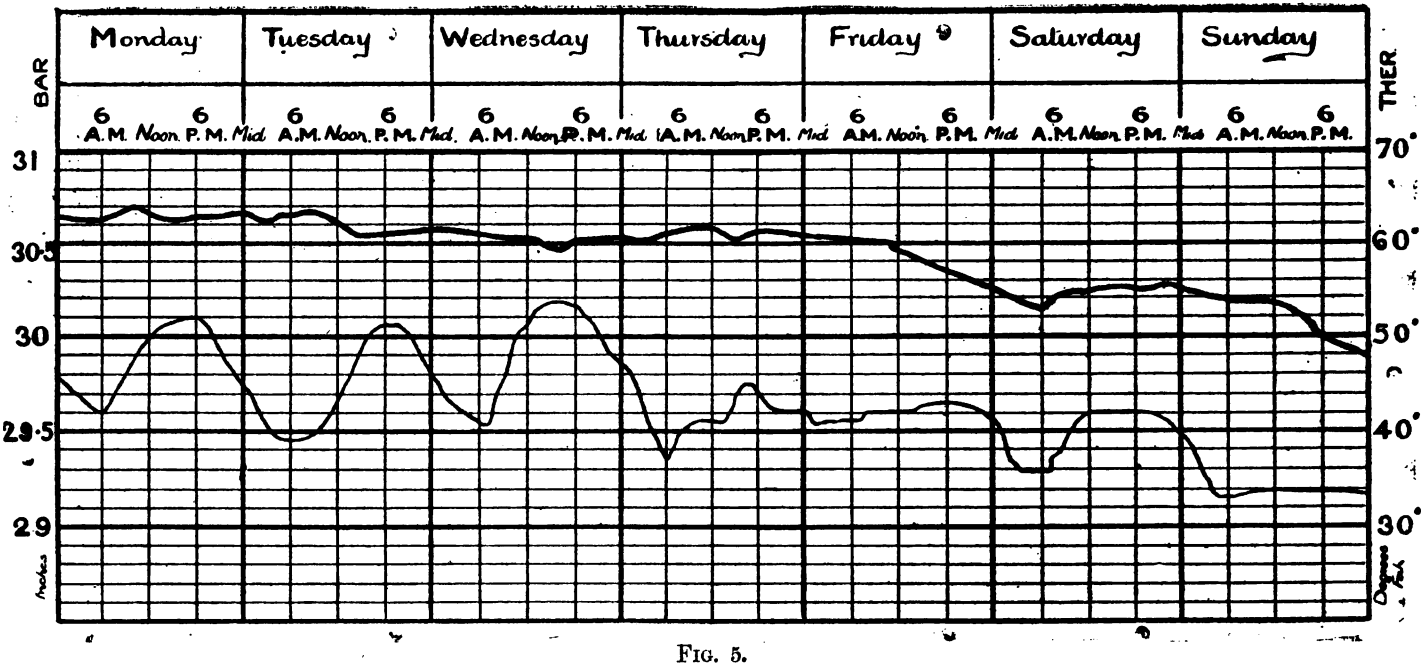
The use of Squared Paper.

25. A sheet of squared paper is covered with equidistant horizontal and vertical lines. Every tenth line is very distinct, so that it is easy to measure off horizontal and vertical distances without using a scale. The paper has its scales on it everywhere, in fact.

Before 1876 sheets of squared paper were very expensive; they were only used by a few people in important work. In that year Prof. Ayrton and I began to use it extensively in Japan, and when we returned to London and introduced at the Finsbury Technical College our methods of teaching Mechanical and Electrical Engineering and laboratory work which have now become so common, we saw that one essential thing was the manufacture of cheap squared paper. It can now be bought for 7½d. a quire instead of 8d. per sheet. Our students treat it almost like scribbling paper. This year the candidates in three important subjects of the Science and Art Department will, for the first time, write their answers upon books of squared paper. It is of importance that the student should use many sheets of squared paper, use them lavishly. It used to be that many men knew how squared paper *might* be used, but they really never used it, or if they did use it, they used it not for solving problems but for illustrating methods of solving problems.

26. In this and my next lecture I mean to show you some of the uses of squared paper. It would be easy to divide this subject up into 150 propositions and lead you on from one to the next, and so build up a science; but here at the very beginning, I want you to understand that, just as I said when describing the slide rule, all the following exercises are really one exercise. A student ought, after doing one or two of them, to see the general idea underlying them all, and if he will only practise by himself and exercise his common sense, he will be able to solve any such problem, and furthermore, he will need no elaborate proofs, things will be so self-evident as to require no proof.

Proofs indeed! Some people need proofs that they themselves exist. The mathematicians will tell you that this subject ought to be called Co-ordinate Geometry or Analytical Geometry. They will tell you that nobody ought to be allowed to begin it until he has mastered the most elaborate Algebraic and Trigonometric investigations. Now


Monday
Tuesday
Wednesday
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BAR
THER
6 A.M. Noon 6 P.M. Mid
31
30·5
30
29·5
29
Inches
70°
60°
50°
40°
30°
Degrees Fah.

FIG. 5.

I want you to understand that I have known it to be used, and used very wisely and well, by a man who could neither read nor write.

27. I have here, Fig. 5, cut from this morning's newspaper, this squared paper record of the rise and fall of the barometer and thermometer for the last week. Do you not all understand at once the meaning of this? Horizontal distances represent time since Sunday midnight. Vertical distances represent heights of the barometer in the one case, and heights of the thermometer in the other.

28. Trades' newspapers have many records on squared paper. Here are some squared paper records taken from the engineering papers showing fluctuations in the prices of some of the metals.

29. Here, again, are some curves showing the output of coal and of iron year by year since 1878 by Britain and America, Germany, France, Belgium, &c., which I find on exhibition in this museum. They show at a glance what you want to know. People interested in coal and iron will read an interesting story in every little fluctuation which you see. The general rate of growth of the industries is evident; what is most striking being their enormous development in America.

30. Think of a silk merchant in Yokohama putting on record in this way the price of silk per pound in Italy as it is telegraphed to him. Why does he do it? First, he has a record of the price for years back; a record read at a glance; a record showing at a glance the times when the price reached a maximum or a minimum value; times when the market was disturbed. Second, he sees by the *slope* of the curve the rate of increase or fall in price. Third, if he plots other things on the same sheet of paper at the same dates, he will note what effect their rise and fall have upon the price of his silk. Fourth, he gets so much information from his curve that he is able to prophesy with more certainty than a man who has no such records; indeed, he may actually be able to say with some certainty what price his silk will sell for in Italy a month hence if he now sends a consignment.

Practical money-making men and philosophers may use squared paper for so many useful purposes that I shall not attempt to enumerate them.

31. I once read a clever article in the *Nineteenth Century*, by one of our greatest statesmen, concerning the rates of

increase of the English population and wealth. The reasoning was most abstruse. On taking the author's figures, however, and plotting them on squared paper, every result which he had reasoned out so elaborately was plain upon the curves, so that a boy could understand them.

32. I was once sitting on a committee when a manager was detailing reasons why attendance at certain classes was

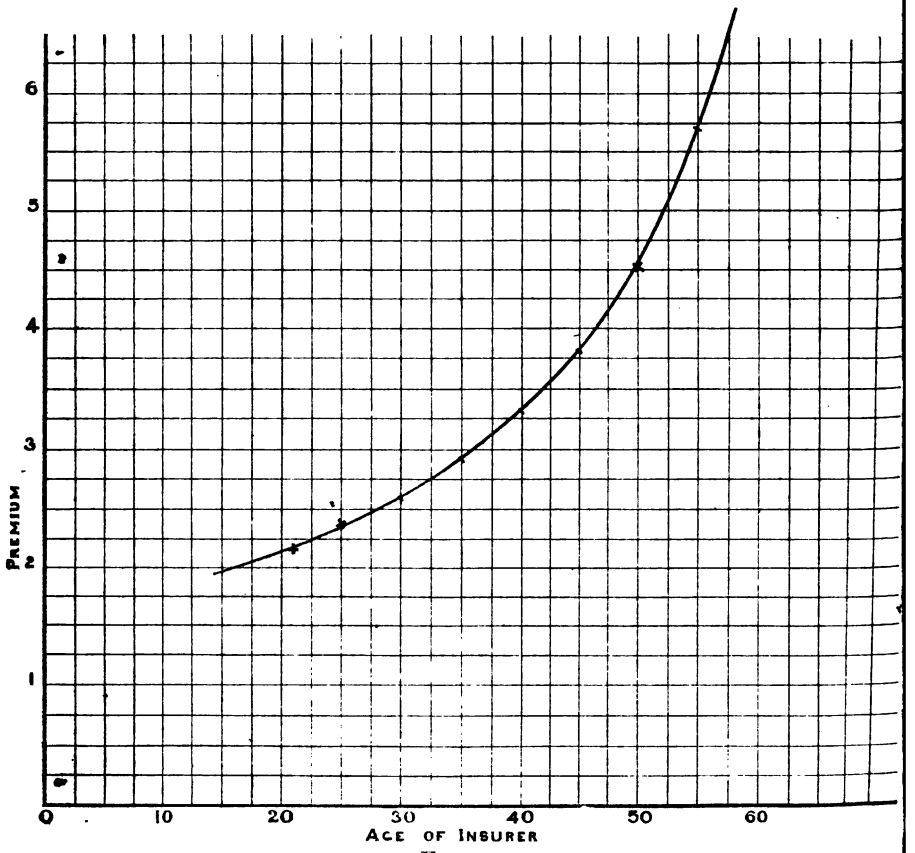


FIG. 6.

steadily dropping. In idleness I manufactured some squared paper, quite roughly, and plotted the numbers, and it became at once evident that some curious event had happened at a particular date which had produced the mischief. This led at once to a rectification of the evil. Now I do not say that a man clever at figures would not have discovered this from the figures themselves, but the

importance of the squared-paper method of working is that no worry over details of figures distracts one from the general story told by them.

33. A student ought at once to use squared paper for himself, and use it with numbers in which he is interested. Many of you are in the watch trade. Well then, there are numbers published concerning your trade which, if you plot them on squared paper, will show you the rate and the yearly change of rate at which your trade is leaving Clerkenwell and going to America, and Birmingham, and Coventry, and the Continent. If this is too sad a subject, borrow a *Whitaker's Almanack*.

You will see a table showing the price of Consols every year since 1790. Plot it as a curve, and get some notion in this way of the decreasing value of money. Or, take this:—

Ex. A certain insurance office gives assurance of £100 at death for the following yearly premiums—

Age of Insurer	21	25	30	35	40	45	50	55	60
Premium ..	£ s. d. 2 3 1	£ s. d. 2 6 6	£ s. d. 2 11 9	£ s. d. 2 18 2	£ s. d. 3 6 3	£ s. d. 3 16 4	£ s. d. 4 10 7	£ s. d. 5 13 8	£ s. d. 7 4 9

You had better convert the shillings and pence into decimals of a pound.

Thus take £5 13s. 8d. Here 8d. is $\frac{8}{12}$ or 0·667 of a shilling, and 13·667 shillings = ·6834 pounds. So I would use £5·6834.

Now having drawn a curve, Fig. 6, through your plotted points; note that you can *interpolate*—that is, you can make a very good guess at the premium which would be charged an insurer of any intermediate age. Besides, the curve itself will teach you a good deal by its mere appearance.

You might, in the same way, take an exercise from the table showing the price of an annuity for a person of a particular age.

You will find an excellent exercise in the table showing the average length of life which may be expected by persons of a particular age.

Plot the total amounts of gold or silver, produced from mines every year since 1887. Draw the curve that lies most evenly among the points. See how nearly you can prophesy the amount of production during this coming year.

Plot the total numbers of letters and of postcards posted every year since 1885 in two curves. Note if one of them shows a peculiarity at any time whether the other was sympathetic.

Plot the increasing revenue or expenditure in India, or the increasing commerce of the United States; or the emigration from England; or the value of imported mutton or apples every year since 1885.

Ex. (2). The following are the numbers of half-time children in schools in twelve months following September 1, in each of the following years:—

Year - - -	1892.	1893.	1894.	1895.	1896.
Attendance - -	164,018	140,831	126,896	119,747	110,654

Plot these and prophesy what the attendance will be in the following year. Also produce backwards and say what the probable attendance was from September 1st, 1891, to September 1st, 1892.

[This exercise was given because just then there was a Bill before the House of Commons dealing with the half-time system in schools.]

Plot the traffic receipts or the number of passengers of the railways of the United Kingdom.

Show in a curve how the number of second class passengers has diminished since 1883, and the number of third class passengers increased.

I mention just a few exercises which one notices on picking up a book like Whitaker's Almanack.

34. *Ex.* (3). The following numbers give some population statistics in millions; let us say in the middle of each specified year:—

——	1811.	1821.	1831.	1841.	1851.	1861.	1871.	1881.	1891.
England and Wales	10·164	12·000	13·897	15·914	17·928	20·066	22·712	25·974	29·002
Scotland - - -	1·806	2·091	2·364	2·620	2·889	3·062	3·360	3·736	4·026
Ireland - - -	—	6·802	7·767	8·175	6·552	5·799	5·412	5·175	4·705

Draw curves, Fig. 7, passing evenly through the plotted points. Do such exercises as these: (*a*) What was the probable population of England and Wales in 1845? (*b*) What was

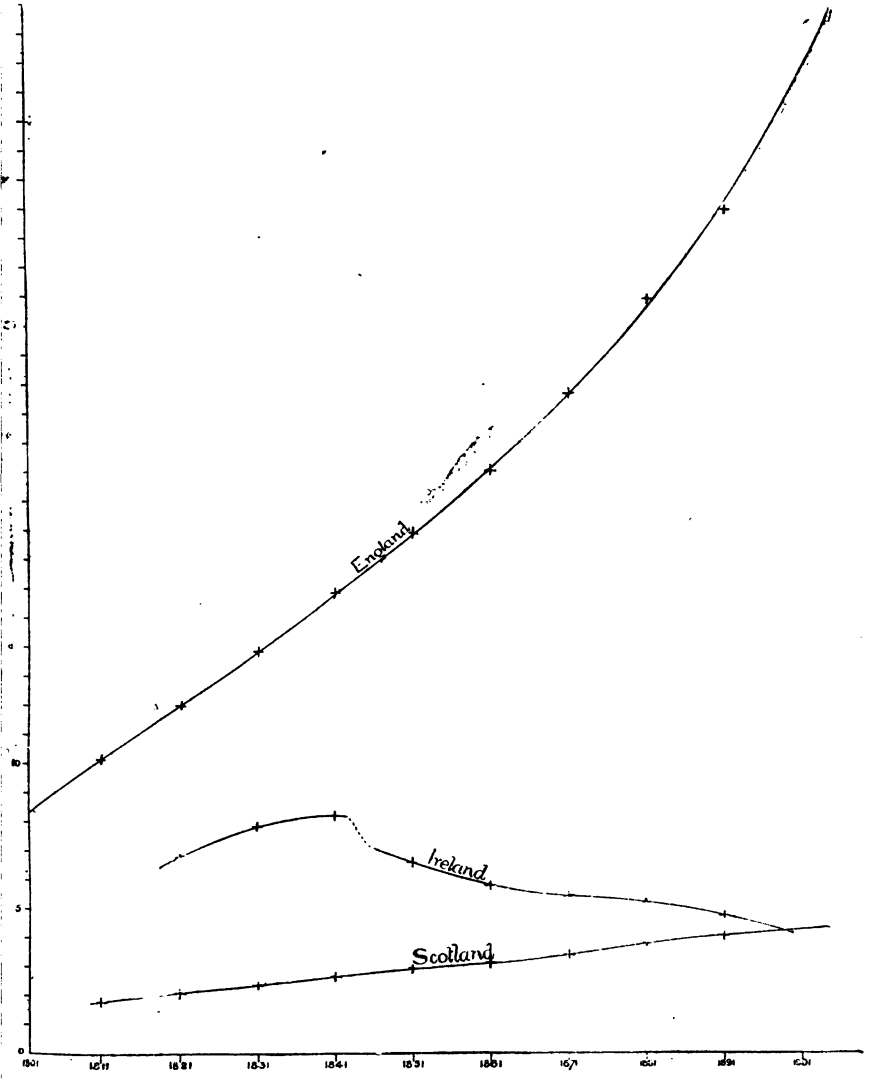


FIG. 7.

the probable rate of increase per annum in the middle of 1845? (*c*) What will be the probable population of England and Wales in 1901?

The student may here be troubled by my asking him for a rate of increase per annum at a particular time. He could give me the actual increase of population during the year 1845. The other idea is really familiar enough to him, but we must consider it in Art. 79.

In exercises like this students will notice that although the numbers may be perfectly exact, we may let our curve pass, not exactly through the points, but only evenly among them, if we are trying to see if there is some simple general law of growth of population.

Ex. (4). The following measured numbers are taken from a certain table very useful to engineers. I want to know p when Θ is 152. Also I want to know Θ when p is 75.

θ	140	145	150	155	160	165
p	52·52	60·40	69·21	79·03	89·86	101·90

By plotting on squared paper [and a student ought to use a large sheet; some candidates in examinations waste a square inch of paper when they ought to waste a square foot of it] you can interpolate.

Answers $\Theta = 152, p = 73{\cdot}1$; $p = 75, \theta = 153$.

The student will notice that although the above numbers are derived from experiment, and are therefore probably slightly in error, yet as they have already been corrected for errors, his curve ought to go exactly through the plotted points

NOTE.—Another way. Suppose we know that certain tabulated numbers follow a certain simple law without great error from one end of the table to the other, the law may be used for very exact interpolation.

Example.—The above values of θ (temperature of steam) and p (pressure of steam in pounds per square inch) are known to follow this law sufficiently well for interpolation :

$$p = a(\theta + b)^5$$

where a and b are constants.

Given the following values, $\theta = 150, p = 69{\cdot}21$; $\theta = 155, p = 79{\cdot}03$, find a and b and then calculate p when θ is 152, also θ when p. is 75.

[Methods of interpolating and finding rates of increase by tabulating differences need not be taken up in this elementary treatment of the subject. They are not difficult to understand].

35. In all the above work, when I ask a student to prophesy, what are the things that I have not warned him about, not told him about? I have not told him to exercise

his common sense. I have not told him that in crossing the street he is in danger of being knocked down by a cab. Of course I could spend hours in talking about self-evident things! But as for the philosophy of the idiosyncrasy of cab-drivers, it is too large a subject. It is even worse than Euclid.

Ex. (5). A student has made experiments and tabulates his observations as follows. Thus he found that when x was 80 his y was 0·55. Never mind now what his actual x or y was. His x was perhaps the temperature of steam and y was its pressure. Or his x may have been ampères of electric current and y may have been volts of potential difference between some two places. In any laboratory experiment we are always finding how one thing depends upon another; x may be ampère turns on a magnet and y the magnetic field produced; x may be the gallons of water per second flowing through a water meter and y may be the angular motion of a pointer on a dial, and we are going to graduate that dial from our experimental results. Anyhow, the student knows that there are errors in his measurements. He will plot the numbers on squared paper; he will try to make some simple curve pass evenly among the points. Thus he will find the probable errors of his observations. If they seem to be too great he will probably reform his method of experimenting.*

x	80	100	150	190	250	300
y	·55	·78	·97	1·10	1·22	1·24

Now let a student take the above numbers and do as requested. He ought to know that he may use any scales whatever, and hence he had better use such scales as will give him points not merely in one corner of a sheet of paper. He will also notice that he may as well plot not the whole of y or of x, for there are no values of y less than ·55, nor of x less than 80. I get the curve shown in Fig. 8. I make the probable errors of the above observed values of y to be respectively—

·013, ·03, −·03, ·01, ·02, −·01.

The − sign means that I think the observed y too small.

* If he finds that they are greater than he thinks they ought to be this is the beginning of a new kind of investigation, for it may be that his assumption that the curve is a simple one is wrong.

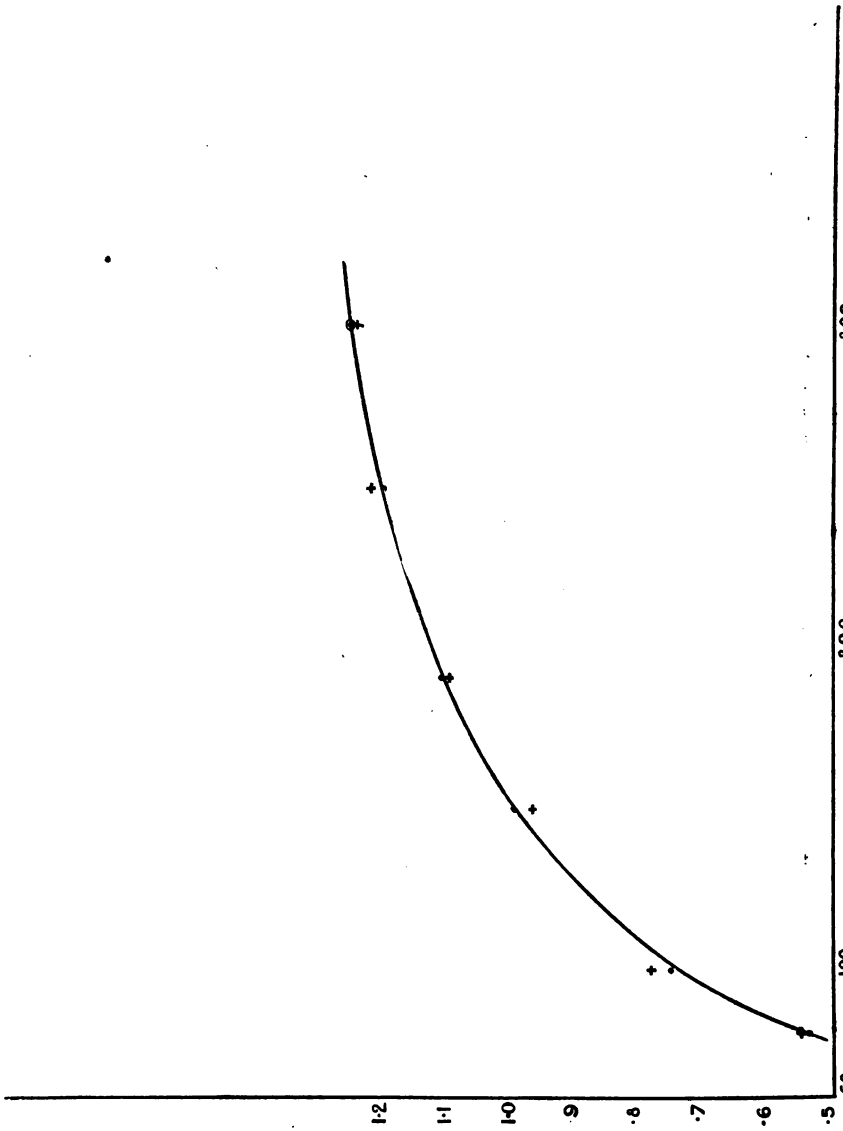


FIG. 8.

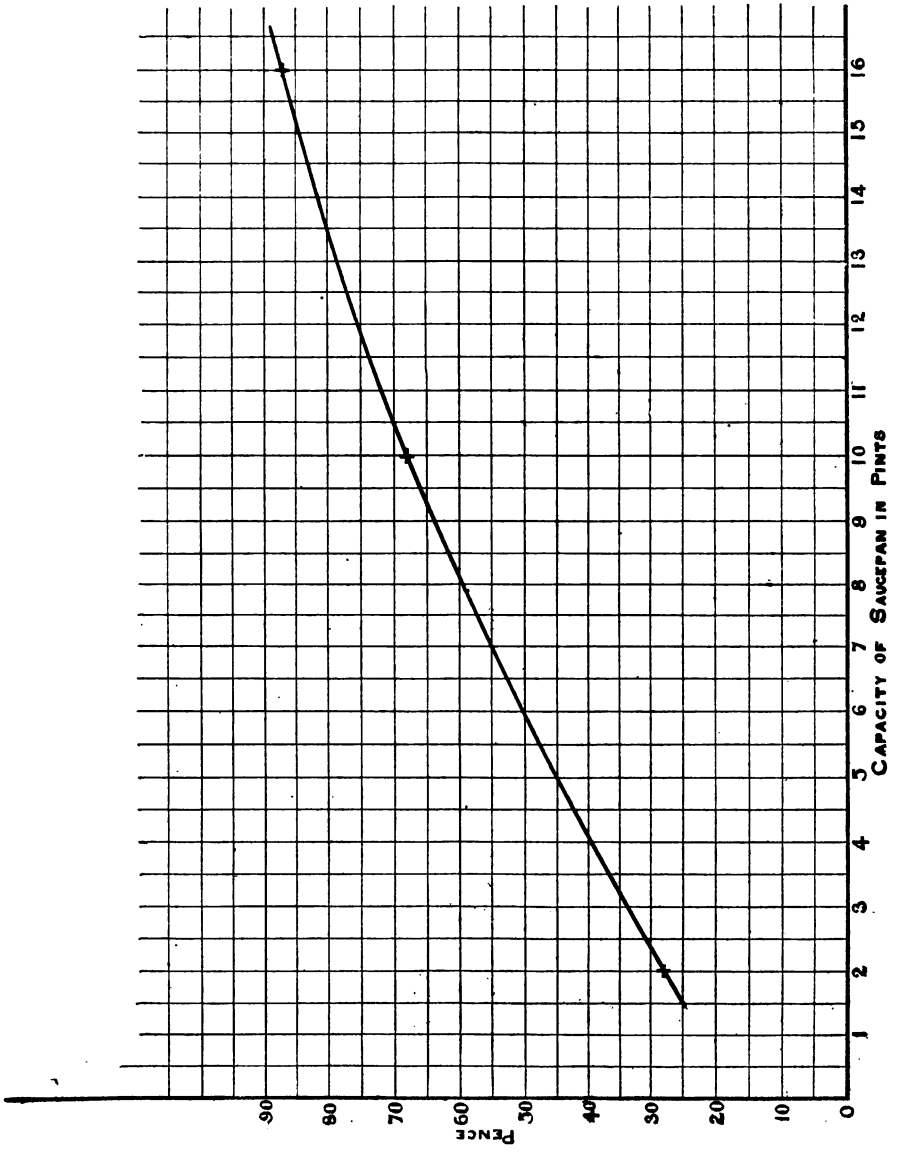

Capacity of Saucepan in Pints
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
Pence
10
20
30
40
50
60
70
80
90

Fig. 9.

Ex. (5*a*). In the above case, y was not observed when x was 170; what is its probable value? I find 1·06.

36. *Ex.* (6). A man makes saucepans. He has only made them of three sizes, as yet, but he knows that other sizes will be wanted. He wished to **publish a price list** of many sizes. I don't know much about the saucepan trade, but suppose he has fixed on the following as really correct prices from every point of view:—

A 16 pint saucepan, price 87 pence.
A 10 pint saucepan, price 68 pence,
A 2 pint saucepan, price 28 pence.

Now let him plot these sizes and prices on squared paper and join his points by a curve, Fig. 9, say by bending a straight-edge. Any point on the curve shows size, and probable best price, of a saucepan.

Thus I make the best price of a $1\frac{3}{4}$ gallon saucepan to be 81 pence, and of a 1 gallon saucepan to be 60 pence.

37. Many manufacturers merely find carefully the price of two sizes of thing, and plotting these properly, they use a straight line as giving the prices of all other sizes.

Ex. (7). A man has made the following sizes of a certain type of small steam engine. and arranged their prices very carefully.

Horse-power 4, price £44.
Horse-power 12, price £108.

Plot these on squared paper. Join the points by a straight line, and make out the probable prices of other sizes.

Thus I get, for 10 Horse-power, the price £92.

38. When we are dealing with tables of numbers that are quite correct, our curve must pass exactly through the plotted points if we are going to interpolate. Thin battens of wood may be bent, weights resting on them here and there to enable a line to be drawn. Sometimes a bent straight-edge is found to do well enough. The following is an exercise exceedingly interesting, not merely as **illustrating interpolation,** but as a continuation of what I have already said about logarithms.

Ex. (8). **To calculate a Table of Logarithms,** or of Anti-logarithms.

Some friends of mine assert that no man or boy ought to be allowed to use logarithms until he knows how to calculate them. They say this knowing that the calculation is a branch of Higher Mathematics, and that the average

schoolboy after six years at mathematics finds it hopeless to even begin the study of the Exponential Theorem. It is a hard saying! It is exactly like saying that a boy must not wear a watch or a pair of trousers until he is able to make a watch or a pair of trousers. It is the sort of unfeeling statement which so well illustrates the attitude of the superior person.

Having recently discovered an easy way of calculating logarithms [I find that it is described in a book published about two days ago by Mr. Edser, an Associate of the Royal College of Science, and he has the priority over me as an inventor], and seeing that it is a very good illustration of our present work, I do not think that there can be any harm in giving it here.

I assume that a boy can extract square roots by arithmetic. Let him, then, extract the square root of 10, and the square root of this again, and so on. Thus he finds $10^1 = 10$, $10^{\frac{1}{2}} = 3{\cdot}1623$, $10^{\frac{1}{4}} = 1{\cdot}7783$, $10^{\frac{1}{8}} = 1{\cdot}3336$, $10^{\frac{1}{16}} = 1{\cdot}1548$, $10^{\frac{1}{32}} = 1{\cdot}0746$.

From these by multiplication he can find $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$, $10^{\frac{7}{32}}$, &c.

Now $\frac{1}{16}$ is the logarithm of 1·1548; $\frac{1}{8}$ is the logarithm of 1·3336.

Let him use ·125 instead of $\frac{1}{8}$; in fact, let him use only decimals, and he has a table of which I give here only the beginning, the middle, and the end.

Logarithm.	Number.
0	1·0000
·03125	1·0746
·06250	1·1548
·09375	1·2409
·12500	1·3336
.	.
.	.
.	.
·46875	2·9427
·50000	3·1623
·53125	3·3982
.	.
.	.
.	.
.	.
·90625	8·0584
·93750	8·6596
·96875	9·3057
1·00000	10·0000

If now he wants the logarithms of numbers between, say, 3 and 3·4. Let him plot three points on squared paper, using the whole of his sheet.

Log.	Number.
·46875	2·9427
·50000	3·1623
·53125	3·3982

Join these points by a curve; a slightly bent straight-edge enables this to be done very nicely. He will now be able to read off any logarithm of any number between 3 and 3·4, or any number for any logarithm between ·47 and ·53. In this way, even with cheap squared paper, he can calculate

the tables, correct to 4 figures. If he wants greater accuracy he must use $10^{\frac{1}{12}}$.

39. Some **Mensuration** Exercises. Before we take up the following exercises, it will be well to tell the student how we usually find the area of an irregular figure.

Simpson's Rule. Divide the area into any even number of parts by an odd number of equidistant parallel lines or ordinates, the first and last being possibly of no length, for they must touch the boundary curve. Take the sum of the extreme ordinates (in many cases 0), four times the sum of the even ordinates, and twice the sum of the odd ones (omitting the first and last); multiply the total sum by one-third of the common distance asunder. This will give the area nearly.

The reason for this rule is given by the integral calculus There are others which are more or less accurate. A less correct but simpler rule which is very **commonly used** is this: divide the area into any number of parts by equidistant parallel lines, the first and last touching the bounding curve. Midway between every two measure the breadth of the figure; add up these breadths and divide by the number of them; call this the average breadth. Multiply by the length or perpendicular distance between the extreme lines to get the area. In indicator diagram work we usually take ten parts.

A **planimeter** (usually *Amsler's*) is an instrument which enables us to measure the area of a figure in square inches or square centimetres or other units. It is very valuable when one has many areas to find. We let the tracing point go round the boundary of our area exactly to the starting point; we can start anywhere; the increase in the dial reading gives the area. The principle of the Amsler planimeter is not difficult to understand, and ought to be given to students. When a curve has many loops there is an interesting rule as to whether the area of a particular part ought to be called positive or negative. We need pay no attention to any such rule when working with the planimeter. Hence it is that the planimeter is so valuable when we require the average pressure on a gas or oil engine indicator diagram. We find the area by the planimeter, and divide by its extreme length parallel to the atmospheric line.

40. Just as we find an area having equidistant ordinates, so we can find a volume if we have areas of equidistant parallel sections, using either Simpson's rule or the other rule I have given.

Ex. 9. A reservoir with irregular sides has the following dimensions. When filled with water to the vertical height h feet above a datum level, the following is the area A square feet of the water surface. The height of the lowest point above datum level $h = 20$.

h	20	25	30	35	40	45	50	55	60	65	70
A	0	11,800	23,600	37,100	51,000	61,500	76,010	89,000	102,000	118,250	130,300

Find the volume in cubic feet when filled up to $h = 70$.

Answer $3{\cdot}177 \times 10^6$.

41. Centres of Gravity.—We often speak of the centre of gravity of a body, or of an area, or of a curve, when we mean centre of inertia, or centre of area, or linear centre.

1. Multiply each small portion of mass m of a body by its distance x from a plane; indicate the sum as Σmx. Divide this by the whole mass of the body M or Σm, and we have the distance of the centre of inertia or mass of the body from the same plane.

Make this calculation for three planes, and you get the exact position of the centre of mass.

If a body is symmetrical about a plane, one-third of the labour is saved. If the body is symmetrical about an axis, two-thirds of the labour is saved.

We write our rule as

$$\Sigma mx = \mathrm{M}\bar{x}$$

2. Multiply each small portion a of an area by its distance x from a straight line in the same plane. Indicate the sum as $\Sigma\, ax$. Divide this by the whole area A or Σa, and we have the distance of the centre of area from the line. Make this calculation for two lines, and you get the exact position of the centre of area.

42. *Ex.* (10.)—Let F B G C, Fig. 10, be an irregular figure. To find its area and the position of its centre of area, draw FD and OG, two parallel lines touching the extreme boundary, and let D A O be any line at right angles to these. Divide D O into many equal parts and draw ordinates at the middle of each.

The sum of all such ordinates as B C divided by the number of them is the average breadth, and this multiplied by D O is the area. Or we may write it if d is the distance A Q of the ordinates asunder ;—

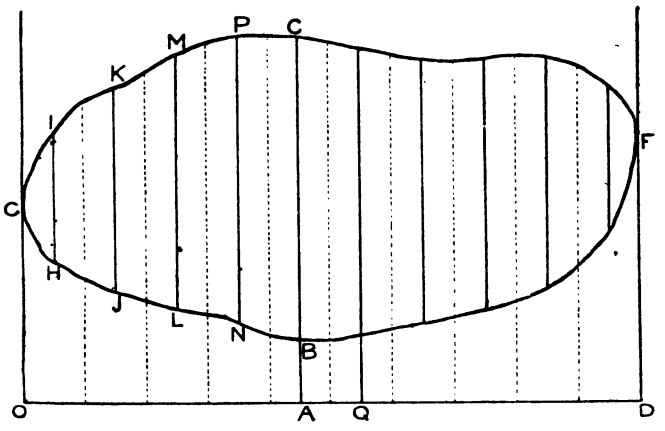


Fig. 10.

$$\text{Area A} = d\,(\text{H I} + \text{J K} + \text{L M} + \text{N P} + \text{B C} + \&c.),$$

Also, if O X is the horizontal distance of the centre of gravity of the area from O G, by the definition

$$\text{O X} = \frac{d\cdot\text{H I} \times \frac{1}{2}d + d\cdot\text{J K} \times 1\frac{1}{2}d + d\cdot\text{L M} \times 2\frac{1}{2}d + \&c.}{\text{whole area}}$$

This is evidently

$$\text{O X} = \tfrac{1}{2}d\,\frac{\text{H I} + 3\cdot\text{J K} + 5\cdot\text{L M} + 7\cdot\text{N P} + \&c.}{\text{H I} + \text{J K} + \text{L M} + \text{N P} + \&c.}$$

Another pair of boundary lines like O G and D F must now be chosen and the work repeated before we can find the actual position of the centre.

43. *Ex.* (11.)—There is a homogeneous body symmetrical about an axis, the following are its areas of cross section A at the distances x inches from one end. Find its volume, and its centre of gravity. As it is homogeneous I shall take volume to mean mass in the calculation. The whole length is 200 inches.

x	10	30	50	70	90	110	130	150	170	190
A	320	304	311	297	292	279	287	274	263	251

Answer.—The sum of the values of A is 2,878, so that the average section is 287·8 square inches. This multiplied by the whole length is 287·8 × 200 = 57,560 cubic inches.

To find the centre of gravity. We imagine slabs measuring 20 inches axially for each given section. Find, then, the sum

$20 \times 320 \times 10 + 20 \times 304 \times 30 + 20 \times 311 \times 50 +$ &c.

This is evidently $200(320 \times 1 + 304 \times 3 + 311 \times 5 +$ &c.) or 5,528,800.

This divided by the whole volume is 96 inches, the value of x of the centre of gravity.

44. *Ex.* (12.) An irregular area A B C D, Fig. 11, has the following breadths measured at right angles to the direction O M, which we choose, at random, to call the direction of its length:—

The lines O A and C M are at right angles to O M and touch the figure at A and C.

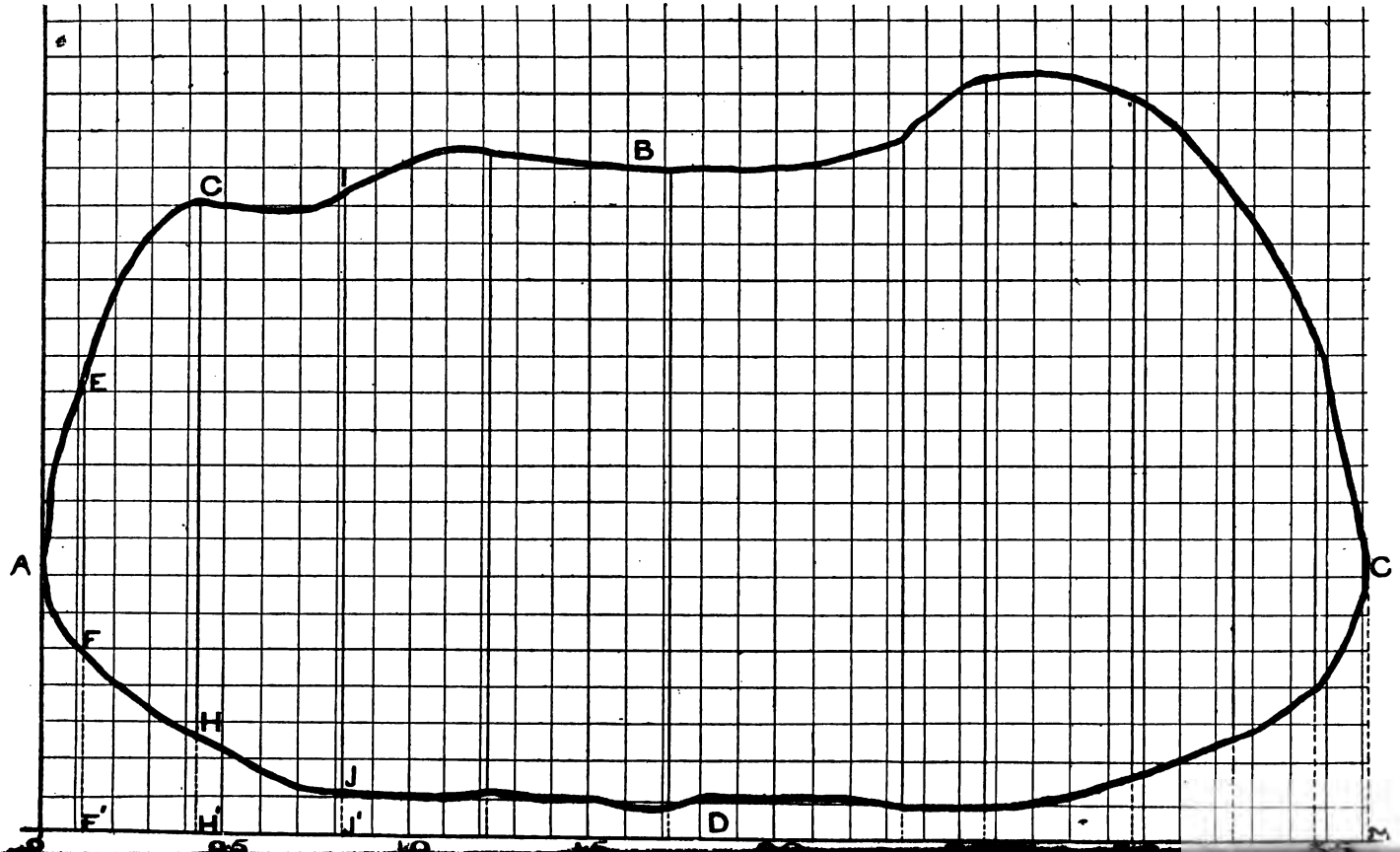
I shall call I J a breadth of the figure, and the perpendicular distance of I J from A (or O J′) I shall call its x. The following measurements have been made:—

Breadths in inches.	0	·75	1·45	1·62	1·73	1·71	1·78	1·95	1·82	1·47	0·95	0
Corresponding values of x in inches	0	·123	0·426	0·823	1·22	1·72	2·34	2·57	2·97	3·25	3·47	3·62

Suppose these numbers to be given, but no other information about Fig. 11. Of course, if the figure itself were given, we should not need to proceed in the following way:—

(1.) Plot the breadths and values of x on squared paper and draw a curve as shown in Fig. 12, where, for example, O J represents x = ·823, and the ordinate J P represents the breadth 1·62.

This curve will enable us to get other breadths. For example, we can get breadths for equal increments of x and use Simpson's rule. Or we may find the area of O P M by the planimeter, and this is the area of the original figure. Or we may take the less accurate method employed with Indicator diagrams.


A
B
C
C
D
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J'
M

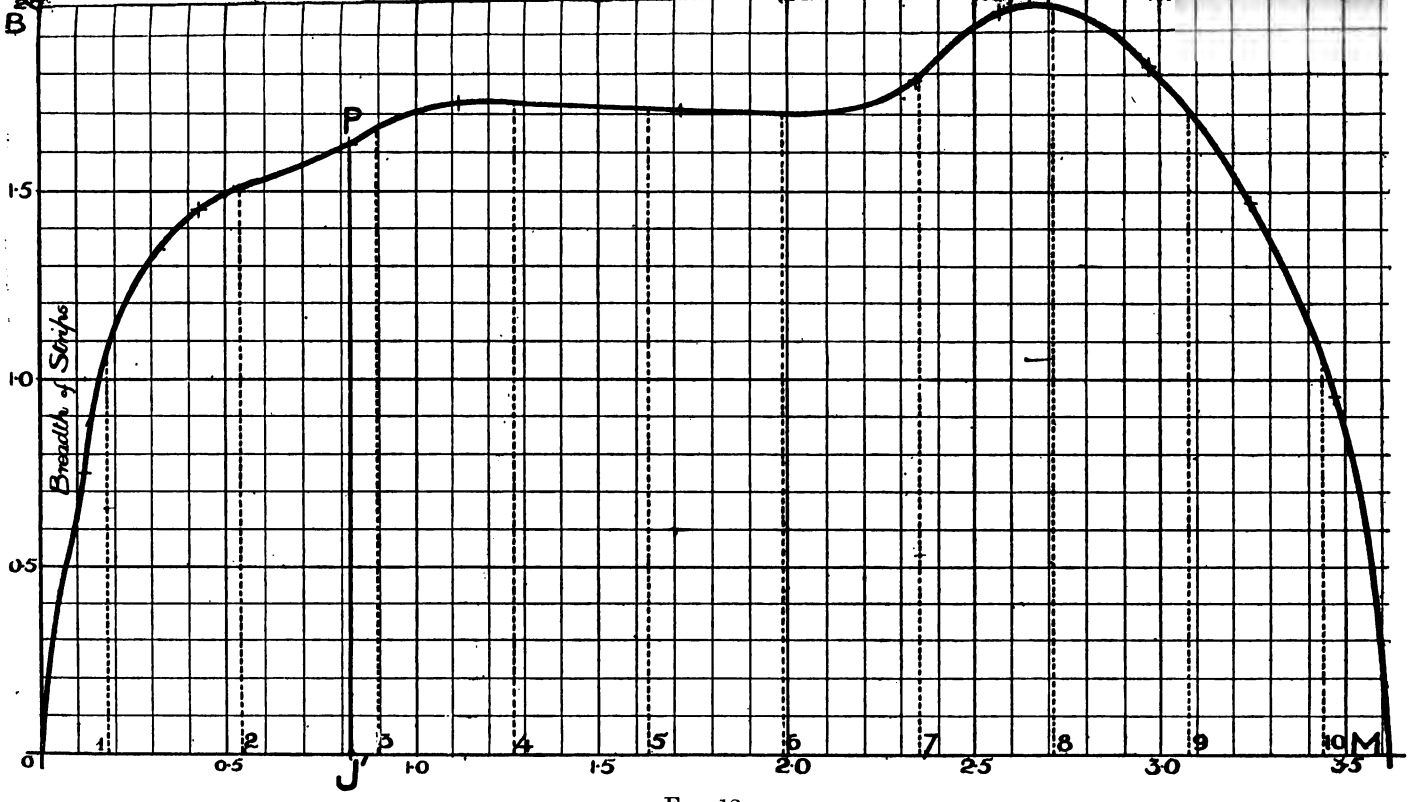

B
P
1·5
1·0
0·5
0
Breadth of Strips
1
2
3
4
5
6
7
8
9
10
M
0·5
1·0
1·5
2·0
2·5
3·0
3·5
J'

FIG. 12.

(2.) The distance of the centre of area from the line *OA* Fig. 11, is the same as the distance of the centre of area of O P M O Fig. 12 from the line O B.

Thus on dividing O M into 10 equal parts, and measuring ordinates at the middles of each, I find the following—

x	·181	·543	·905	1·267	1·629	1·991	2·353	2·715	3·077	3·439
Breadth - - -	·96	1·53	1·65	1·74	1·71	1·72	1·80	1·97	1·70	1·30

Now the sum of these is 16·08, and dividing by 10 we have the average breadth 1·608. The length OM being 3·62 inches the area is $1{\cdot}608 \times 3{\cdot}62 = 5{\cdot}82$ square inches.

By the rule of Art. 42 we find the distance of the centre of area from OA to be 1·88 inches.

45. *Ex.* (13.) The following are the areas of cross section of a body at right angles to its straight axis—

A or Area of cross-section in square inches.	0	75	145	162	173	171	178	195	182	147	95	0
x inches the distance of the section from one end.	0	12·3	42·6	82·3	122	172	234	257	297	325	347	362

Plot A and x on squared paper. The average value of A is easily found to be 160·8. This multiplied by the whole length 362 is $362 \times 160{\cdot}8 = 5{\cdot}82 \times 10^4$ cubic inches, the volume. The average value of A is, of course, found in the same way as in the last case.

The centre of gravity of the body is found to be 188 inches from the end from which x is measured. It corresponds to the centre of area in position.

46. *Ex.* (14.) Find the volume of a reservoir when its greatest depth is 42 feet, given the following areas, A square yards of the surface of water when this surface is h feet vertically above the lowest point of the bottom.

A	0	2,100	8,200	13,100	15,500	19,500	25,400	32,400	47,100	52,000
h	0	5	10	17	21	25	29	33	38	42

Plotting A and h on squared paper and taking the average height of the curve, either by Simpson's rule or the less accurate method or the planimeter, we get the average A to be 20,000 square yards or 180,000 square feet. This multiplied by 42 gives 7,560,000 cubic feet, the answer.

47. *Ex.* (15.) One man on the front of a tram car looks at a well-damped spring balance inserted between the horses and the draw bar, thus measuring the pull on the car in pounds. I call this pull F. Another man has a means of measuring x, the number of feet passed through from some mark on the line. A third man has a watch and notes the time t seconds that have elapsed from some arbitrary time. They make simultaneous observations of F, x, and t, which are given in this table.

F	650	630	615	585	540	510	460	450	450	500	550
x	500	600	750	870	950	1,100	1,300	1,400	1,500	1,650	1,800
t	0	10	21·71	29·37	34·02	42·46	53·44	59·03	64·78	73·69	82·85

1st. Plot F and x on one sheet of paper, and find the average value of F.

2nd. Plot F and t on another sheet of paper, and find the average value of F.

It is worth your while to think of the reason why these averages are not equal.

48. *Ex.* (16.) Give on one sheet curves which show roughly how the number of radians and the sine, cosine and tangent of an angle alter as the angle alters from 0° to 90°. Fig. 13 shows the result.

Thus O G represents 90°, O B represents 50°, B C shows to scale 0·6428 the cosine of 50°, B S is the sine of 50° or 0·7660, B R is the radians of 50° or 0·8727; B T is the tangent of 50° or 1·1918. O R^1 is straight. O S S^1 droops downwards and is horizontal at S^1. O T T^1 goes up to infinity. C^1 C G is exactly like O S S^1 as seen in a looking glass.

When a student knows how to find the sine, etc., of angles, not merely between 0° and 90°, but of all sizes, he ought to plot the above curves, say from − 360° to + 360° as an exceedingly interesting exercise.

49. *Ex.* (17.) The *average height* of the positive part of a sine or cosine curve, is the fraction $\frac{2}{\pi}$ of the greatest height or *amplitude*. Test this.

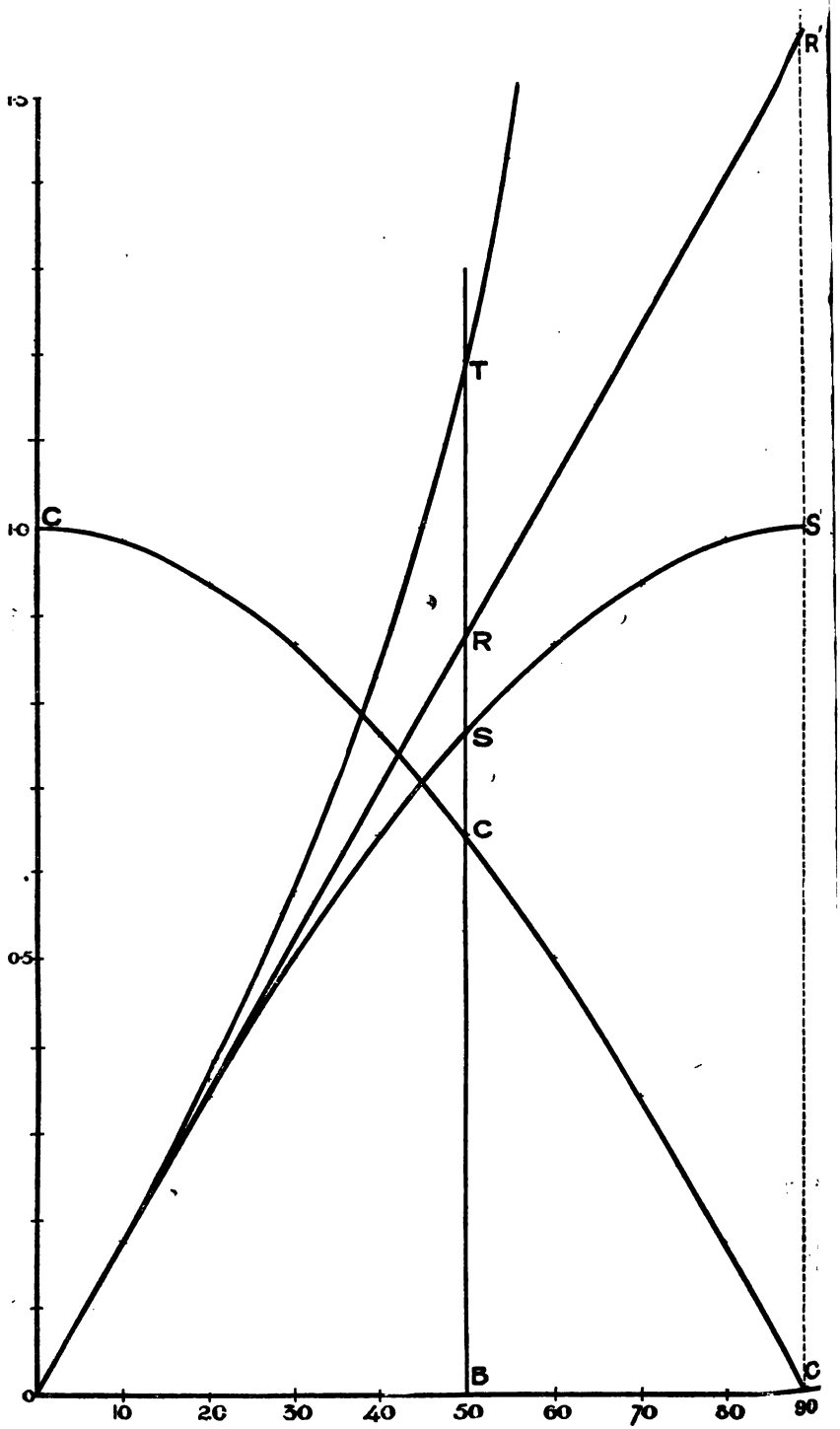


FIG. 13.

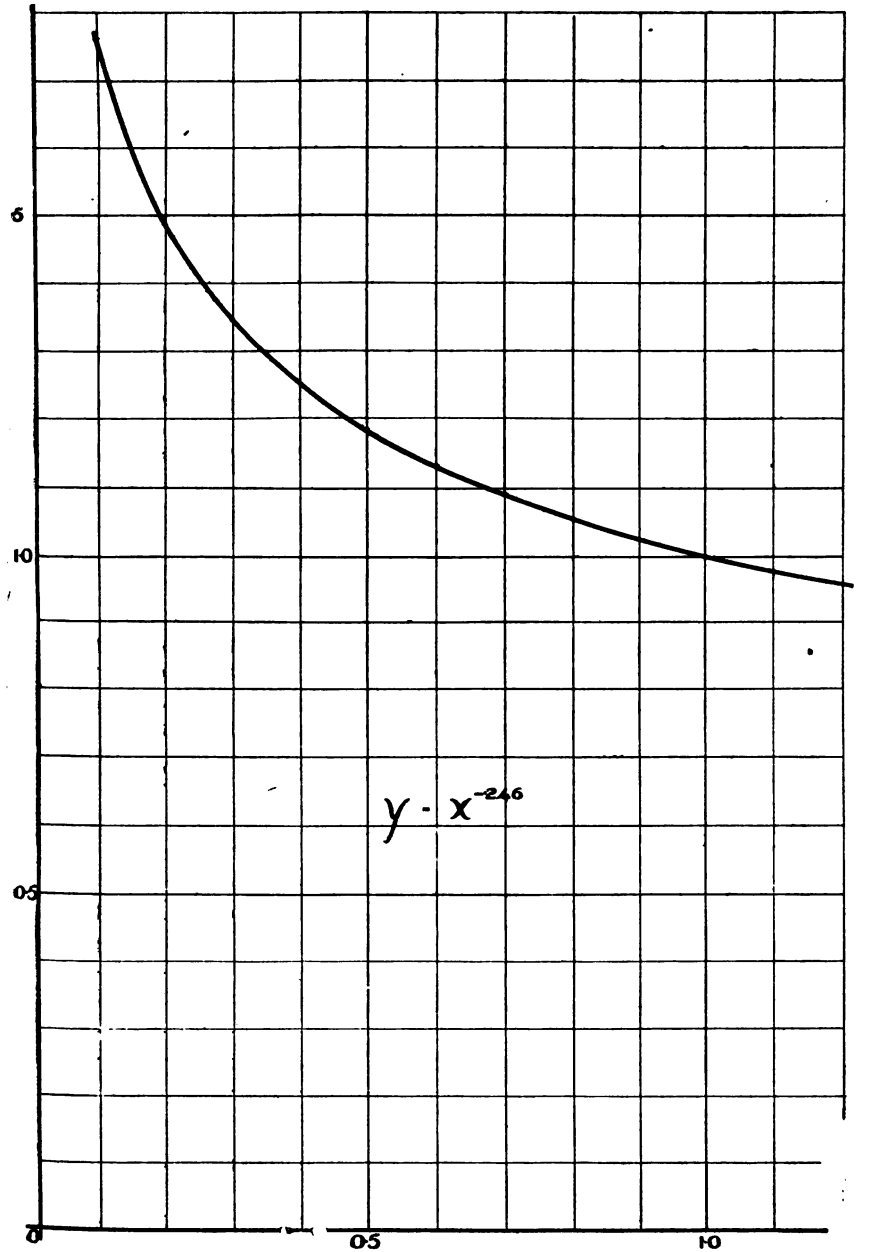


FIG. 14.

LECTURE IV.

Squared Paper (continued.)

50. When one quantity, say y, is expressed as an algebraic function of another which I shall call x; take any value of x and calculate the corresponding y; plot on squared paper; draw the curve which passes through many such plotted points.

The most important functions to plot are—

(1) $y = ax^n$ where a and n are any numbers whatsoever.

Thus $y = 9x$ or $y = 2x^2$ or $y = 3{\cdot}5x^3$, or $y = 10x^{\frac{1}{2}}$,

or $y = 5x^{\frac{1}{3}}$ or &c.,

or $y = 9x^{-1}$, which may be written $xy = 9$,

or $y = 9x^{-2}$, or $y = 4x^{-3}$ or &c.,

or $y = 2x^{-0{\cdot}246}$ or $y = 2x^{0{\cdot}246}$.

Perhaps it might be well to use 1 as the value of a in every case so that students in a class may draw the whole amily of curves producible by using different values of n. It is good to prick them through on one sheet of paper.

(a) As an example I will show the plotting of, say

$$y = x^{-0{\cdot}246}.$$

Taking $x = {\cdot}1$, I find $y = ({\cdot}1)^{-0{\cdot}246} = 1{\cdot}762$,

„ $x = {\cdot}2$ „ $y = ({\cdot}2)^{-0{\cdot}246} = 1{\cdot}486$,

and in fact by taking the following values of x I get the following values of y:

x	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0	1·1
y	1·762	1·486	1·345	1·253	1·186	1·133	1·091	1·056	1·027	1·000	·977

I give this in Fig. 14.

(b) Find the average value of y between $x = {\cdot}3$ and $x = 1{\cdot}1$.

Answer. Measuring the average breadth in the usual way, I find that the average y is 1·114.

The area then of the curve between $x = {\cdot}3$ and $x = 1{\cdot}1$ is $1{\cdot}114 \times 0{\cdot}8 = 0{\cdot}891$.

In $y = b + ax^n$ . . . (1)

we see that b is a mere constant addition; a merely determines the scale of measurement; hence if we study

$$y = x^n \quad . \quad . \quad . \quad (2)$$

we may be said to have a complete study of (1).

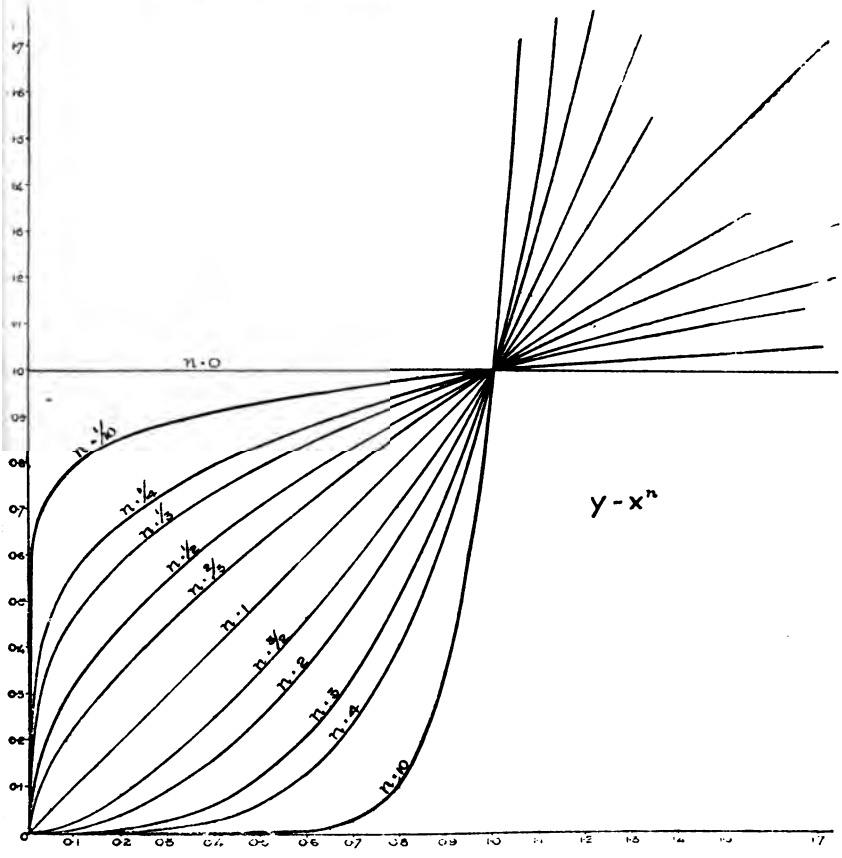


FIG. 15.

Now let a number of students get together; let each of them take one curve and plot it. When all are finished let them be pricked through upon one sheet of paper. Curves with various values taken for n are shown upon Fig. 15. They may be said to form one family. I gave a separate set of drawings, Fig. 16 for the negative values of n.

(2.) Study the family of curves represented by

$$y = ae^{bx} \quad . \quad . \quad . \quad (1).$$

As before, a may be taken as 1, and the family of curves

$$y = e^{bx} \quad . \quad . \quad . \quad (2)$$

may be drawn upon one sheet by students, each of whom

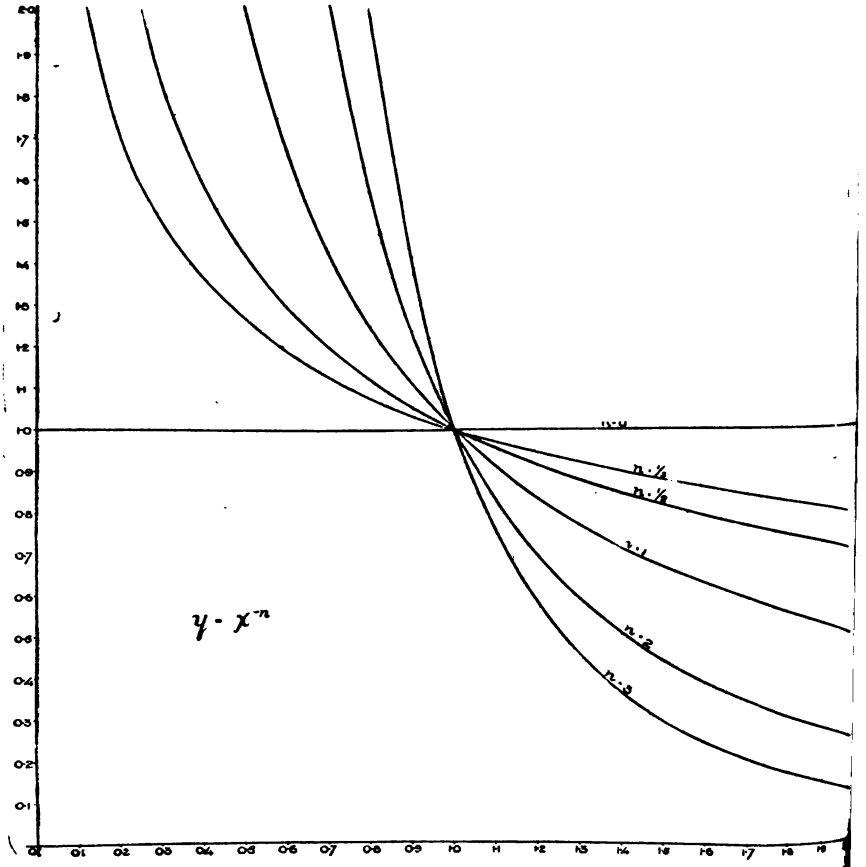


FIG. 16.

takes a particular value of b. The work is easy if one has a table of Napierian logarithms, because

$$\log_e y = bx \quad . \quad . \quad . \quad (3).$$

Or if one has only a table of common logarithms,

$$\log_{10} y = 0{\cdot}4343\ bx \quad . \quad . \quad . \quad (4).$$

Curves with various values taken for b are shown, Fig. 17.

The man who invents a very quick method of finding an where a and n are any numbers whatever, say by means of a slide rule, or by the use of a table, will confer a great benefit upon students of steam and gas engines.

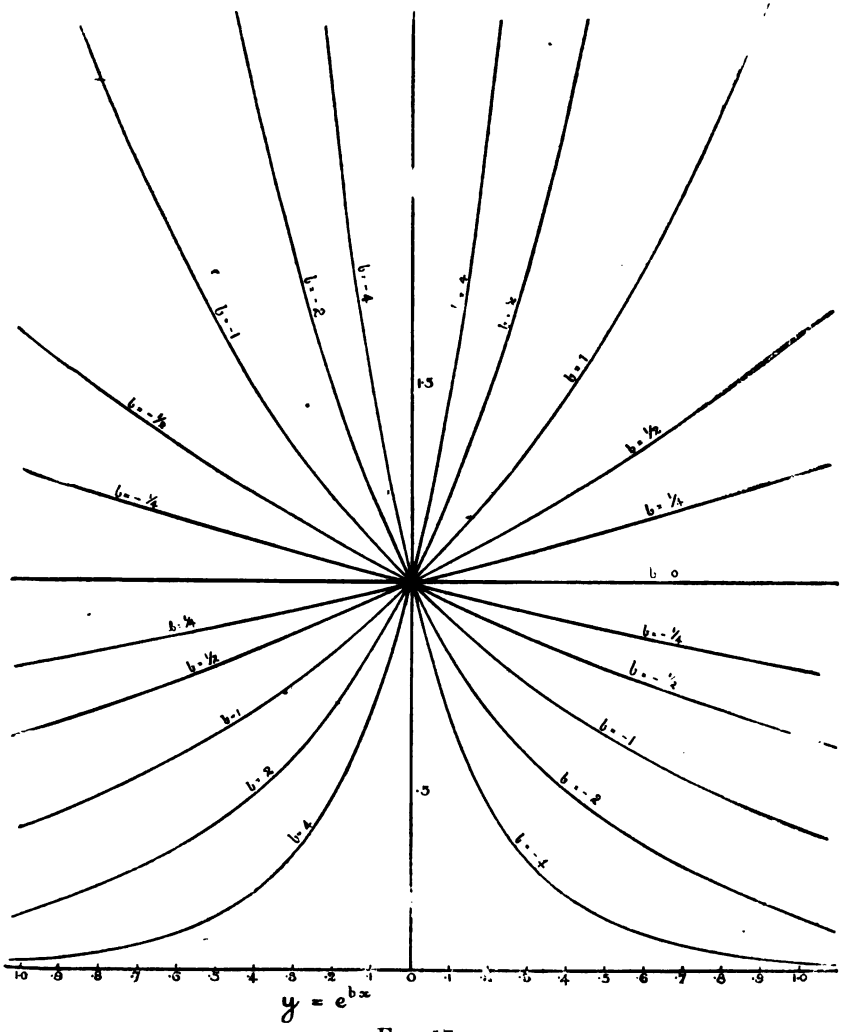


FIG. 17.

51. The Position of a Point.—The following story is not true. During the seven years' war, in 1760, in Saxony, a gentleman buried treasure in four different places on his estate. He was suddenly killed. His son did not know

where the treasure was buried; he knew that it was buried. But there was a parchment document which his father had confided to him containing these symbols,

$x = 2{,}000$, $y = 977$; $x = -560$, $y = 700$; $x = -750$, $y = -650$; $x = 356$, $y = -274$.

He sought for the treasure in vain.

In 1860 a descendant, a young American, found his way to the old estate, and was welcomed as a distant cousin. He was not very rich, yet he fell in love with a girl cousin as poor as himself. He had long known the old legend about the buried treasure—indeed, no member of the family ever forgot it. He happened to pick up a school-book one day and saw in it this figure, Fig. 18, which he had never seen before,

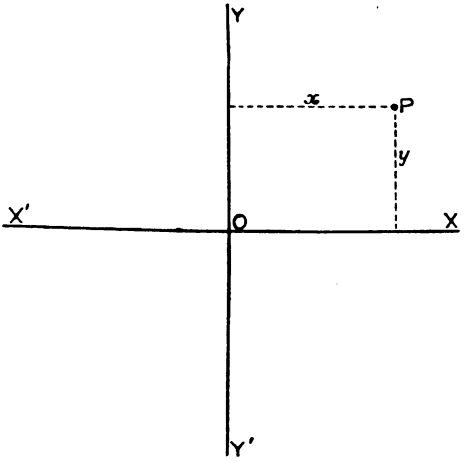


FIG. 18.

albeit, it is very common in mathematical books. It is so very common that no German who has gone to school from the age of seven to the age of 25, and had himself stuffed like a Strasburg goose with mind-training knowledge nine hours a day all that time, would ever dream of connecting it in any way with legend or romance, or, indeed, with anything but a school-book. But to this young American it told a story. Here were x's and y's evidently distances from two well-marked straight lines. So he and his cousin looked up an old 1760 map of the estate, and sure enough they saw two faint lines drawn in continuation of the sides, of the very same old building in which they then were, something like Fig. 19. Then George W. went to Dorothea's

father and said, "If I discover the old hidden treasure will you let me marry my cousin?" "Here are two discoveries," said old Heinrich, "one by you and one by me. I know well that an American, who has no such school-book knowledge as our Germans are hampered with, has imagination and ability to do things, and he is not dull; and therefore I will let you marry Dorothea, even if you have made a mistake in

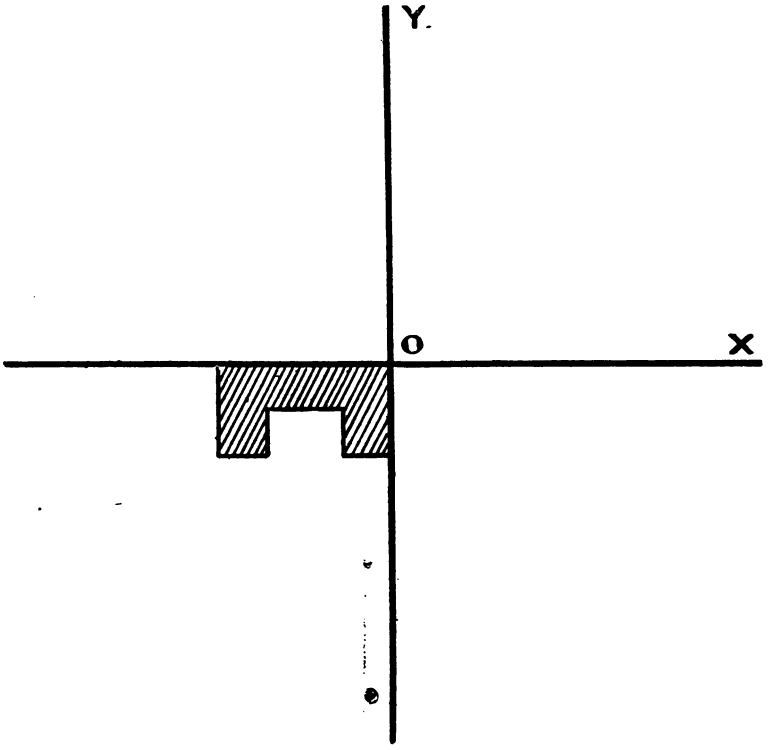


FIG. 19.

this thing." Then did George Washington Ollendorff make this sketch, Fig. 20, for his uncle.

"What the distances are in," he said, "feet or yards, I do not know; but you know what measure they used here a hundred years ago, and the places I have marked are the spots to dig for buried treasure."

Then did old Count Heinrich von Ollendorff turn wrathfully upon the guileless George W. and say,

"Now do I see that you have been deceiving me all this time, for you know about Cartesian Co-ordinate two

dimensional Analytical Geometry, and although nobody agrees with me on this matter, I affirm that you must be dull and stupid."

But Dorothea placated her father, and said: "You know, dear father, that I am only your little haus fraulein, and know nothing of learning, and yet even I see that if AP is 2,000, being measured to the right of a line, A'P' may well be called − 560, being measured to the left of the same line. Also if PB is 977 being measured above the

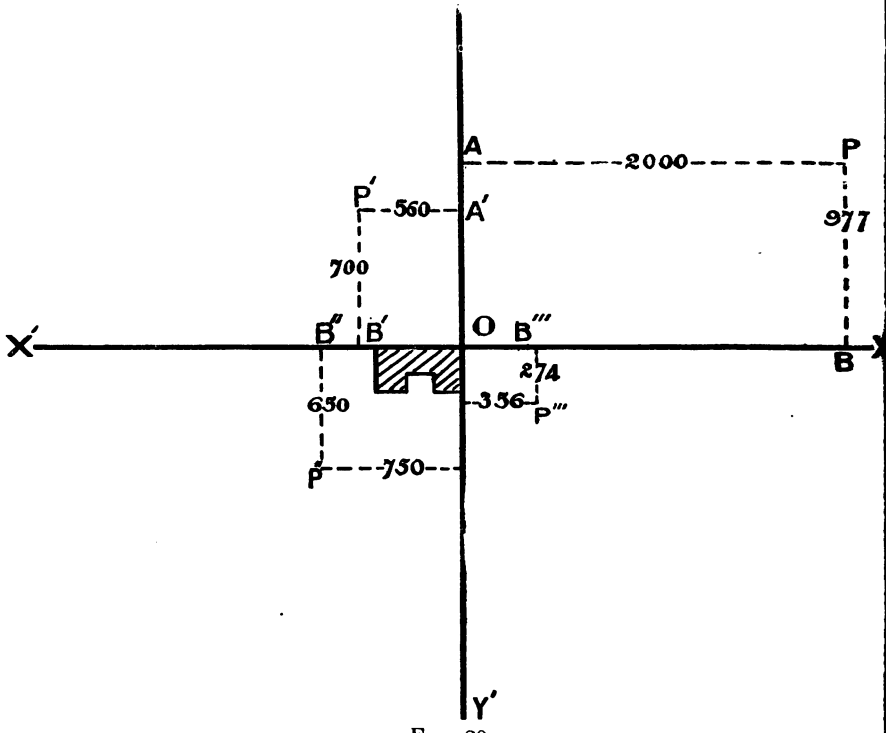


FIG. 20.

line, P'''B''' may well be called − 274, being measured below the line.

Also George W. spoke up, and said, "I guess I know precious little of that terrible science you mentioned just now, but I say that this is a common sense way of bringing together the legendary figures and the legendary map. Besides, I have your promise, and I here declare that for myself I will take none of this treasure; my treasure was buried here at the place of our **origin.**"

But old Heinrich insisted on dividing the four buried treasures among the 207 Von Ollendorffs of Europe and 310 Ollendorffs of America, and the joint shares of George W. and Dorothea amounted to 1·2307 dollars.

52. When a point is not on a flat surface, we must still give two dimensions to find its position. Latitude and longitude are quite familiar to you all; thus, for example, if I say that a small island has been discovered in 56° N. latitude and 30° W. longitude, it is easy to find its place on a map or a globe. A shipmaster can tell his position with a possible error of only a mile or so, if his chronometer is to be depended upon, and if he can see the sun.

53. *Ex.* . Plot the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It is easy to see that this is the same as

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \text{ or } y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

There are then two values of y for each value of x, and again it is evident that $x = +4$ or $x = -4$ will produce values of y that are equal to one another. To make this clear one must work a few exercises.

(1.) Suppose $b = 5$ and $a = 5$ the formula simplifies to

$$y = \pm \sqrt{25 - x^2}.$$

Take $x = 1$ then $y = +\sqrt{24}$ and also $y = -\sqrt{24}$; that is, there are two valves of y for $x = 1$, and so we get two points. Indeed we get four points by the one calculation, for $x = -1$ would give the same answers for y.

Now take $x = \pm 2$ and calculate y and so get other four points, and so on.

(2.) Suppose $a = 5$ $b = 3$.

Then

$$y = \pm \tfrac{3}{5}\sqrt{25 - x^2}.$$

If you have done (1) you see that this will be very easy.

54. *Ex.* Plot the curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is evidently the same as

$$y = \pm \frac{b}{a}\sqrt{a^2 + x^2}.$$

take $b = 3$ and $a = 5$ and so plot the curve

$$y = \pm \tfrac{3}{5}\sqrt{25 + x^2}.$$

55. When a curve is simple looking, it may often be expressed by a simple algebraic formula.

When we plot corresponding values of y and x, and find that the points lie in a **straight line,** we always find that there is a simple law connecting them of the form

$$y = a + bx$$

where a and b are constants.

It is worth while spending some time in plotting this function.

Thus plot $y = 2 + 0{\cdot}75\ x$. By taking the following values of x, I have calculated the corresponding values of y. Let these be plotted on squared paper. You will find that the points lie exactly in a straight line.

x	0	1	2	3	4	5	6	7	8
y	2	2·75	3·5	4·25	5·0	5·75	6·5	7·25	8

I find that a stretched black thread gives the best test of straightness.

Now try

$$y = 2 + 0{\cdot}5x,$$
$$y = 2 + 0{\cdot}9x,$$
$$y = 2 - 0{\cdot}3x,$$
$$y = 2 - 0{\cdot}75x.$$

In every case you will find a straight line.

Now notice that in all these cases we have the same value of a, and consequently all your lines have some one thing in common. What is it? Find out for yourself.

56. Now plot another series of straight lines $y = a + bx$ in which b is common to all.

Take $y = 1 + 0{\cdot}75x$, $y = 3 + 0{\cdot}75x$, $y = 4 + 0{\cdot}75x$, $y = 0 + 0{\cdot}75x$, $y = 1 - + 0{\cdot}75x$, $y = -2 + 0{\cdot}75x$.

You will find all such lines with the same b have the same slope, and indeed I usually call b **the slope** of the line.

57. Slope of a line. If $y = a + bx$. Take $x = x_1$ and find the corresponding y; it is

$$y_1 = a + bx_1.$$

Now take a new x, call it x_2, and find the corresponding y; it is $y_2 = a + bx_2$. Subtracting we get

$$y_2 - y_1 = b\,(x_2 - x_1),$$

or
$$\frac{y_2 - y_1}{x_2 - x_1} = b,$$

or in words,
$$\frac{\text{increase in } y}{\text{increase in } x} = b.$$

Hence whatever values we may take for x_1 and x_2, we find:—

The **rate of increase** of y compared with the increase of x is a constant, b. Now it is the rate of increase of one thing relatively to another which enters into most of our thinking about things, and we notice that this rate is constant in any case, when on plotting the quantities on squared paper we get a straight line.

58. In the laboratory, when we have measured corresponding things, and on plotting we find the points lying in a straight line, we are usually glad, because we know that the things are connected by a very simple law. Besides it is a law which it is very easy to test with a black thread.

Ex. (1). The following observed numbers are known to follow a law like

$$y = a + bx,$$

but there are errors of observation. Find by the use of squared paper the most probable values of a and b.

x	2	3	4·5	6	7	9	12	13
y	5·6	6·85	9·27	11·65	12·75	16·32	20·25	22·33

On plotting, as in Fig. 21, and stretching a black thread o get the straight line which lies most evenly among the points, I find that two points in my selected straight line are

$$x = 2,\ y = 5{\cdot}5\ ;\ \text{and}\ x = 10,\ y = 17{\cdot}5.$$

Hence to make $y = a + bx$ fit these numbers

$$5{\cdot}5 = a + 2b,$$
$$17{\cdot}5 = a + 10b.$$

Subtracting, we have

$$12 = 8b \text{ or } b = 1{\cdot}5,$$

therefore $5{\cdot}5 = a + 2 \times 1{\cdot}5,$

or $a = 2{\cdot}5.$

Hence the law required is

$$y = 2{\cdot}5 + 1{\cdot}5x.$$

The student ought now to calculate y for each of the above values of x, and so see the probable error of the observed values of y.

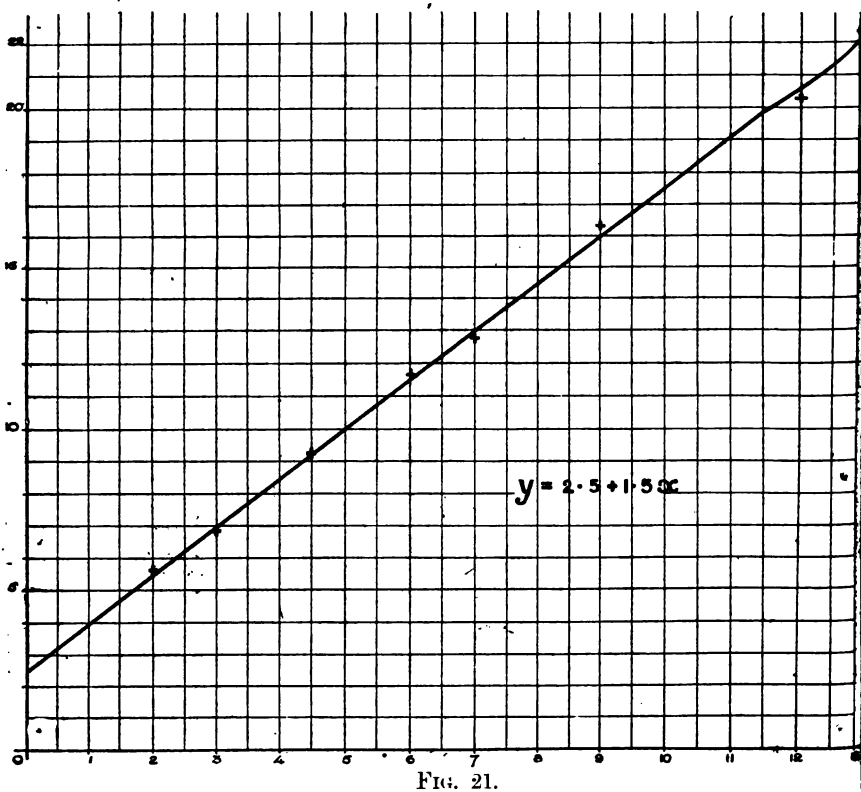


FIG. 21.

Note.—The wooden-headed, cock-sure academic person will tell you that he has an algebraic method of infinite exactness based on the laws of probability for finding the best values of a and b. Do not believe him; the black-thread method is easy to understand, and one therefore has one's wits about one when using it. To the average man using the other method it is occult; his belief in it is like our ancestors' belief in magic. Even the good mathematician forgets that it is based on the assumption that every observation is as likely to be in error as every other one. But with the black thread one cannot help adopting rules as to probability which suit the nature of the observations, especially if one has made them oneself. Very often the probable errors are not all equal, but rather the percentage error or the probable error may be in some curious relation to the observations.

(2). I recently made the following tests of a large single cylinder gas engine. I is the indicated power and B the actual or Brake Horse-power given out by the engine. Plot I and B, and find the law connecting them. I did not measure B when I is 100; if I had made this measurement what, in all probability, would my answer have been?

Brake Horse-power B- -	16	57	95	99	117
Indicated Horse-power I -	35	73	114	120	139

Answer.

$$B = \cdot 96\ I - 16\cdot 4.$$

When $I = 100$ $B = 79\cdot 6$.

(3). If P is the electric power in kilowatts sent out of an electric lighting station, and if C lb. is the amount of coal burnt in the boiler furnaces per hour; find the law connecting P and C if the following tests are correct:—

P	C
349	1,121
291	1,020
228	927
171	820
119	743
71	652

Answer:—

$$C = 1\cdot 67\ P + 540.$$

It is interesting for the student to calculate $C \div P$ in each of these cases, noting the diminution of efficiency of an electric light station when giving out small supplies of power, that is, during the greater part of the 24 hours. It is worth while to draw a curve showing the value of $C \div P$ for each value of P.

(4). When the weight A was being lifted by a laboratory crane, the handle effort B (the force applied at right angles to the handle) was measured and found to have the following values:—

A	0	50	100	150	200	250	300	350	400
B	6·2	7·4	8·3	9·5	10·3	11·6	12·4	13·6	14·5

What law connects A and B?

Answer, $B = 0\cdot 0207\ A + 6\cdot 3$.

Exercise.—The velocity ratio of the above crane being 80, a handle effort $A \div 80$ ought in every case to lift A if there were no friction or other source of unnecessary work done.

Dividing A ÷ 80 by B, that is finding A ÷ 80 B and calling this the efficiency e, plot e and A, and so find a curve showing how the efficiency of the crane depends upon the load.

59. Of course, if we have some theory to guide us it is easy to find the algebraic law connecting x and y. The following are observed quantities:—

x	1·1	1·8	2·5	2·9	3·6	4·3	4·8	5·4
y	1·91	2·13	2·42	2·65	3·09	3·66	4·09	4·73

Plotting on squared paper we find a regular, simple curve. For many purposes this curve is enough; it allows us to correct observations, to interpolate, &c. But if we suspect that there is some simple algebraic law connecting y and x we trust to our experience of curves and to lucky guessing and trial to determine the actual algebraic law.

After trying various things, let us suppose that we are lucky enough to think of plotting y and x^2, or let us suppose that theory tells us to plot y and x^2. Squaring all the values of x we get:—

x^2	1·21	3·24	6·25	8·41	12·96	18·49	23·04	29·16
y	1·91	2·13	2·42	2·65	3·09	3·66	4·09	4·73

I find on plotting now that I get a straight line. As there are evidently errors in the given values of x and y, I stretch my black thread and settle what is the best possible straight line; that is, I fix some two points as being probably quite correct. In this case I fix

$$y = 1{\cdot}9 \text{ when } x^2 = 1$$
$$y = 4{\cdot}3 \quad \text{„} \quad x^2 = 25$$

Hence if

$$y = a + bx^2$$
$$1{\cdot}9 = a + b$$
$$4{\cdot}3 = a + 25b$$

Subtract

$$2{\cdot}4 = 24b, \text{ or } b = 0{\cdot}1$$
$$1{\cdot}9 = a + {\cdot}1, \text{ or } a = 1{\cdot}8$$

Hence

$$y = 1{\cdot}8 + 0{\cdot}1x^2$$

is the law. Now taking the above values of x and calculating y, I can find the probable errors in the observed values.

60. Very often an experienced man who knows a great deal about curves fails to discover a simple law, although it may exist. If the points do not lie approximately in a straight line I often try $y = ax^n$ or $y = ae^{bx}$; but it depends upon the appearance of the plotted curve **what I shall try**.

I remember the first time I discovered a law all by myself. It was in 1875.* I was a very inexperienced young man, and thought I had discovered a law of as much importance as Newton's law of gravitation. Of course I now know that many other empirical formulæ would have suited my numbers equally well; the merit of a formula lies not merely in its fitting the numbers, but also in its simplicity. Any single valued function whatsoever may be represented by a formula like

$$y = a + bx + cx^2 + dx^3 + ex^4 + \&c.$$

if we only take enough terms.†

The merit of Newton's law lies in this, that although it is so extremely simple, it seems to be wonderfully true throughout such space as we have had a chance of applying it to, the space occupied by our solar system. Beyond our system we know nothing about laws of gravitation, but in our usual optimistic way we always assume it to be true, just as we reasonably assume a great many other things to be true which may yet be proved to be false. Indeed we do not know that the law of gravitation as usually stated is true inside our solar system; the weight of a body may really be dependent upon its temperature, for all we know,

* See Proc. Royal Society, 1874-5.

I had observed carefully and found the following results for the conductivity C of glass at varying temperatures θ.

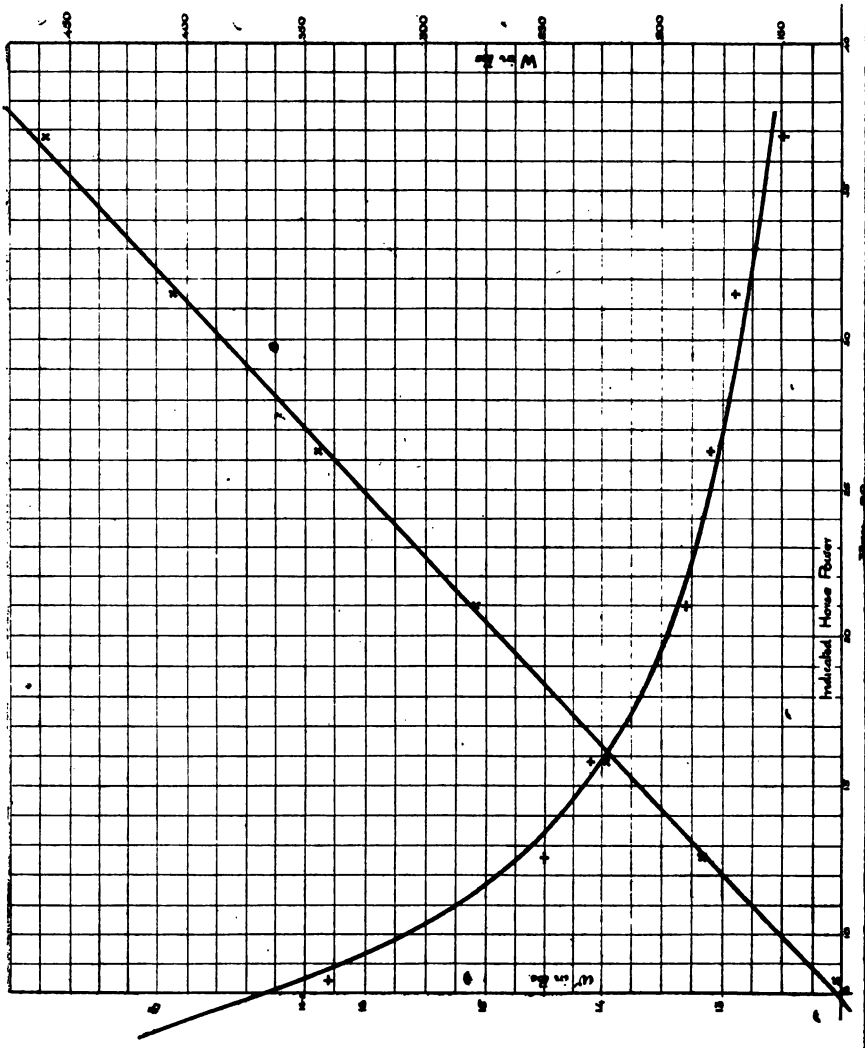
Temp. Fah. θ	58°	86°	148°	166°	188°	202°	210°
C	0	·004	·018	·029	·051	·073	·090

I plotted θ and C on squared paper, and tried in all sorts of ways to find the algebraic law connecting them, for it was evident to me that some simple law did connect them. At length I was lucky enough to think of plotting log. C with θ, and found that my points lay in a straight line, and it was then easy to show that $C = ·000124\ (1·032)^{\theta}$

† At the Royal Society, once, I heard a mathematical man talking about an empirical formula he had discovered. It was a good enough formula, within the range of values of his experiments; but he actually discussed the possibility of unreal values and curious roots of equations altogether out of the region of his experiments, as if he had found an infinitely exact law of nature. The exercise of a little common sense will prevent this kind of "extrapolation."

or some other property that we now assume it to be independent of.

It is astonishing how we go on believing things without proof, and so long as we are aware of the fact there is no harm in it. Hundreds of thousands of lecturers



on chemistry demonstrate the exact composition of the atmosphere to audiences, and then a man of original thought like Lord Rayleigh comes along and really tests these statements and finds them wrong.

61. Now let me give you an example of a **lucky guess.** Here is the sort of result obtained in many thousands of cases of experimenting with steam engines. [I use one of the very best sets of experimental numbers ever obtained from a condensing triple expansion steam engine, tested under seven steady loads, each lasting three hours.]

Indicated Horse Power, I. - - - -	36·8	31·5	26·3	21	15·8	12·6	8·4
Wt. of steam used per hour per indicated horse power, w - - - - - -	12·5	12·9	13·1	13·3	14·1	14·5	16·3

Plot I and w on squared paper, as in Fig. 22, and you will find the sort of result obtained by me and many hundreds of other people year after year. Somehow we could make no use of our results; there was no simple law. But **Mr. Willans** was lucky enough to think of plotting with I, not w, but W the whole weight of steam used per hour by the engine. Now try. Of course I × w is what I call W.

I.	36·8	31·5	26·3	21	15·8	12·6	8·4
W.	460	406·2	344·5	279·3	222·8	182·7	137·0

You see that the points lie sufficiently nearly in a straight line for us to be able to say that

$$W = 37{\cdot}5 + 11{\cdot}5\ I \quad - \quad - \quad - \quad (1),$$

a very simple law. If we use Iw for W and divide by I we get

$$w = \frac{37{\cdot}5}{I} + 11{\cdot}5 \quad - \quad - \quad - \quad (2),$$

and therefore if it had struck anybody to plot not w and I but w and $\frac{1}{I}$, a simple law would have been discovered.*

62. *Example.*—The following values of x and y were observed, and it was important to see if some simple algebraic law connected them.

x	0	·05	·1	·3	·5	1	2	1·4	2·5
y	0	·136	·26	·55	·78	·97	1·22	1·1	1·24

* NOTE.—In every case tried by Mr. Willans he found a linear law connecting W and I. It can be proved that in non-condensing engines it is reasonable to expect the linear law to hold, but that it is not true in condensing engines unless special precautions are adopted to keep the cylinders dry. Also, it is true in steam engines only when we regulate by letting the steam pressure vary; the cut-off is not supposed to be altered.

Plotting on squared paper, you will notice that when x and y are small, we may say roughly that

$$y \propto x,$$

but as x gets greater and greater y seems not to get proportionately greater, but rather to be approaching a limiting value. In any case of this kind first try if

$$y = \frac{ax}{1 + bx}$$

is true, that is if $y + bxy = ax$;
or, dividing by x, if

$$\frac{y}{x} + by = a.$$

This will be tested if we plot $\frac{y}{x}$ and y on squared paper. In the present case I find that the points so plotted do lie very nearly in a straight line such that

$$\frac{y}{x} = -2y + 3$$

and hence

$$y = \frac{3x}{1 + 2x}.$$

63. The following values of x and y when plotted on squared paper give a curve differing in character from the last principally for the smaller values.

x	0	0·5	1·0	1·5	2·0	2·5	3·0
y	0	0·48	1·37	2·12	2·58	2.93	3·07

The student ought to plot them, and then see the reasonableness of trying

$$y = \frac{ax^2}{1 + sx^2}.$$

This may be written

$$y + syx^2 = ax^2, \text{ or}$$

$$\frac{y}{x^2} + sy = a.$$

The law will therefore be very well tested if we plot y and $\frac{y}{x^2}$ on squared paper. If this is done we find a straight line to lie evenly among the points, and I take

$$y = \frac{2{\cdot}2x^2}{1 + 0{\cdot}6x^2}$$

as representing the true law.

64. *Example.*—An engineer wanted to be able readily to state approximately the total cost K of steam plant, that is of the buildings, boilers, pipes, fittings, foundations, and engines for any given maximum indicated horse power I. He obtained actual figures from a number of people who had recently (1896) put up good steam plant under the average conditions existing in and about English manufacturing towns. Taking rough averages, he found—

If I = 200, K may be as great as £4,600 or as little as £3,800; the average may be taken to be £4,200.

If I = 30, K may be as great as £900 or as little as £550; the average may be taken to be £725.

If I = 120, K may be as great as £2,600 or as little as £2,300; the average may be taken to be £2,450.

Plotting on squared paper, I find that a straight line will lie fairly well among the points, and this leads to the following rule:—

$$K = 100 + 20\,I.$$

What is the probable cost of steam plant whose indicated power is 160 ?—Answer, £3,300.

65. *Ex.* Consider the following observed numbers—

x	1·70	2·24	2·89	4·08	5·63	6·80	8·42	12·4	16·3	19·0	24·3
y	320	411	491	671	903	1050	1270	1780	2250	2520	3180

If these are plotted on squared paper we get a curve. The shape of this curve would suggest to me,

1. To plot log. x and log. y, or
2. To plot x and log. y, or
3. To plot log. y and x, or
4. Try other tricks.

Now it will be found in the present case that the first of these plans succeeds. Thus I find—

log. x	·2304	·3502	·4067	·6107	·7505	·8325	·9253	1·0934	1·2122	1·2788	1·3856
log. y	2·5051	2·6138	2·6911	2·8267	2·9557	3·0212	3·1038	3·2504	3·3522	3·4014	3·5024

and on plotting I find that the points lie very nearly in a straight line. In the way already described, I find the most probable line to be

$$\log. y - ·876 \log. x = 2·299,$$

$$\text{or } yx^{-0·876} = 199.$$

66. I may say that plotting log. x and log. y on squared paper is so often successful, especially in gas and steam

engine work, that an old pupil of mine (Mr. Human) induced a stationer to manufacture sheets of **logarithmic paper**, so that one might lay out log. x and log. y without using a table. Here is a sheet of such paper. You will see that the figure 5 is not at the distance 5 inches from the axis, but at a distance proportional to log. 5.

This paper is especially valuable when instead of the law

$$yx^n = C,$$

the law that is most nearly true is

$$(y + \alpha)(x + \beta)^n = C.$$

For after we have plotted y and x upon this curious sheet and we find that no straight line will lie among our points it is easy to study, not the plotted points, but points lying one or two or three divisions to the right or above, or to the left or below them.

67. *Exercise.*—I have found that the expansion curve of an indicator diagram always seems to follow approximately a law

$$p\,v^n = \text{constant}.$$

The following measurements (it is easy to prove that the scales of measurement are really of no consequence if we want to find n) are taken from a gas engine diagram. But at is known that the clearance was not exactly measured, snd hence all the values of v may need the addition or subtraction of a constant number. Find the law and the correction for clearance.

p	39·60	44·7	53·8	73·5	85·8	113·2	135·8	178·2
v	10·61	9·73	8·55	7·00	6·23	5·18	4·59	3·87

Answer. I find that 0·6 must be subtracted from every value of v as the correction for badly measured clearance. In fact I find

$$p\,(v - 0{\cdot}6)^{1{\cdot}346} = \text{constant}.$$

There is no doubt of the usefulness of Logarithmic Paper in working such an exercise.

68. The following are tabulated values of p, the pressure in pounds per square inch of saturated steam; u being the number of cubic feet per pound. It was found that the law

$$p\,u^n = c, \text{ a constant} \quad - \quad - \quad - \quad - \quad - \quad (1)$$

was very nearly true, and as a formula of this kind is very easy to use in calculations it is important to find the best values of n and c within the following range of values:—

p	6·86	14·70	28·83	60·40	101·9	163·3	250·3
u	53·92	26·36	14·00	6·992	4·28	2·748	1·853

If (1) is true, we have

$$\log. p + n \log. u = \log. c.$$

Plotting log. p and log. u I find that a straight line lies very evenly among the points, and, in fact, we may say,

$$\log. p + 1{\cdot}0646 \log. u = 2{\cdot}68.$$

Hence $$p\, u^{1{\cdot}0646} = 479.$$

69. *Exercise.*—For a certain investigation it would be very satisfactory if we could express $\frac{1}{u}$ as a linear function of p; try if this is possible.

Plotting $\frac{1}{u}$ and p on squared paper I find that although there are discrepancies we may take

$$\frac{1}{u} = 0{\cdot}0171 + 0{\cdot}0021\, p.$$

It depends upon the nature of the investigation what value we ought to place upon such a formula. When I use such an approximation I always test the importance of the discrepance.

70. *Exercise.*—Plotting the following numbers we find a regular curve. After various trials I found that on plotting x and log. y on squared paper I obtained a straight line,*

x	5	6·5	8	9·5	10·5	12·0	13·0	14·2
y	41·7	55·0	74·3	101·0	121	161	196	247

and so found that log. $y = 1{\cdot}1955 + {\cdot}0843\, x$

or, $$y = 15{\cdot}69 \times 10^{{\cdot}0843\, x}$$

or, $$y = 15{\cdot}69\, e^{{\cdot}194\, x}$$

* Of course, if there is a theory to guide us our labour is greatly saved. For example, if we know that the rate of increase or diminution of y with regard to x is proportional to y itself it is the compound interest law, and we know that

$$y = ae^{bx}$$

so that we plot x and log. y.

If we know that $\frac{dy}{dx}$ is proportional to $\frac{y}{x}$ we ought to try

$$y = ax^b.$$

If we know that $\frac{dy}{dx}$ is proportional to x we ought to try

$$y = a + bx^2$$

In work like this there would evidently be a great saving of labour if we had paper ruled logarithmically one way and with equal divisions the other way. It is my intention to get some paper of this kind manufactured.

71. *Exercise.* Show from the table of Art. 34 [plotting t and log. P] that, leaving out the census result for 1811, if t is time in years since 1811, the population of England and Wales may be said to have closely followed the law

$$P = 10^{7\cdot03 + \cdot00556\, t}.$$

Calculate P from this formula, find the small differences from it as given by the census returns, and plot these differences as a curve.* What will be probable population in 1901 and 1911? Answers

$$33\cdot93 \times 10^6 \text{ and } 38\cdot58 \times 10^6.$$

72. It will help your comprehension of what I shall say in my next lecture if you will read the following statement regarding speed.

The idea of **velocity** involves two things—the **direction** and the **speed.** When the direction does not alter, we speak of the speed as if it were the whole idea. Find the time in seconds taken by a body to traverse a certain distance measured in feet. This distance divided by the time is called the **average velocity.** Thus, if a railway train moves through 200 feet in four seconds, its average velocity during this time is 200 ÷ 4 or 50 feet per second. If we find with careful measuring instruments that it moves through 20 feet in ·4 second, or through two feet in ·04 second, the velocity is 20 ÷ ·4 or 2 ÷ ·04 or 50 feet per second. It is important to remember that the velocity may be always changing during an interval of time, however short. To get **the velocity at any instant** we must make very exact measurements of the time taken to pass

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* It is to be remembered that we have had three kinds of problem:—

1. Plotting on squared paper exact numbers calculated from simple formulæ. Our curve goes exactly through the plotted points, and it is a simple curve.
2. Plotting observed numbers: there being errors of observation, we assume that the simple curve going most evenly among the points gives us the correct law.
3. Plotting correct numbers, as of Population, the true curve goes exactly through the plotted points. But there is possibly a simple law complicated by perturbations; in studying the curve which goes evenly among the points we look for the general law. Having it, we search for the perturbation law, if there is one.

over a very short distance, and even this will only give us the average velocity during this short time. But if we make a number of measurements, using shorter and shorter periods of time, the average velocity becomes more and more nearly the velocity which we want. Thus after 10 o'clock **a man in a railway train** making a careful measurement finds that the train passes over 200 feet in the next four seconds. He finds the average speed for four seconds after 10 o'clock to be 200 ÷ 4 or 50 feet per second. Another man finds that it passes over 100·4 feet in the two seconds after 10 o'clock, and finds during these two seconds an average velocity of 100·4 ÷ 2 or 50·2 feet per second. Another man finds 50·25 feet passed over in one second after 10 o'clock, which shows a velocity of 50·25 feet per second. Another man finds 25·132 feet passed over in half a second after 10 o'clock, and finds 25·132 ÷ 0·5 or 50·264 feet per second. Another man finds 12·567 feet in a quarter-second after 10 o'clock, and his observation gives 50·268 feet per second, and so on. It is evident that the values given by these various observations are approaching the **real value** of the velocity at 10 o'clock. Tabulating these results we have:—

Intervals of Time in seconds after 10 o'clock.	Average velocity in feet per second.
4	50·00
2	50·20
1	50·25
½	50·264
¼	50·268

Plot the two sets of numbers on squared paper, and draw a curve through the points so found. Produce the curve, and we have the means of finding the average velocity for an indefinitely small interval of time after 10 o'clock. This is the required velocity (*See* Art. 81).

73. Acceleration.—This is the time rate of change of the velocity of a body. Thus it is known that the velocity of a body falling freely in London:—

At the end of	one second	is	32·2	feet	per second.
„ „	two seconds	is	64·4	„	„
„ „	three „	„	96·6	„	„
„ „	four „	„	128·8	„	„

and we see that there is an *increase* to the velocity of 32·2 every second. The acceleration in this case is always of the same amount, hence we call it *uniform acceleration*, and say it is 32·2 feet per second per second.

74. *Exercises.*—(1). One mile per hour; also, one knot; convert each of these into feet per minute and feet per second. Ans. 88, 1·467; 101·3, 1·689.

(2). A torpedo-catcher travels at 32 knots; convert this into English miles per hour. Ans. 36·85.

(3). Prove that 60 miles per hour means ·0268 kilometres per second.

(4). Ten miles per hour; state this in feet per second, and in centimetres per second. Ans. $14\frac{2}{3}$; 447.

(5). An acceleration of 32·2 feet per second per second; state this in miles per hour per second; state it in centimetres per second per second. Ans. 21·96; 981·4.

(6). Two hundred gallons of water per minute; how many pounds per second? How many cubic feet per second? Ans. 33·3; ·535.

(7). A round pipe six inches diameter has 30 gallons per second flowing through it. What is the velocity? If the diameter becomes 10 inches, what is the velocity? Calculate in the two cases the kinetic energy of one pound of water, this being the square of the velocity divided by 64·4 Ans. 24·5 feet per second; 8·8 feet per second; 9·3 foot pounds; 1·2 foot pounds.

(8). Two fine wires are 10 feet apart; a **bullet** breaks them both. The breaking of each wire causes an electric spark to make a mark underneath a fixed platinum pointer on a revolving drum. If the drum is 4 feet in diameter, and revolves at 300 revolutions per minute, and the spark-marks are found to be 1·32 feet asunder on the curved surface, assuming that the intervals of time between the breaking of the wires and making the marks were the same, find the time between the breaking of the wires and find the velocity of the bullet.

The surface velocity is $\frac{300 \times 4\pi}{60}$ or 62·83 feet per second; 1·32 divided by this gives ·02101 seconds; dividing this into 10 feet gives 476 feet per second as the velocity of the bullet.

Exercise.—In some **gun experiments** screens 150 feet apart were cut by a bullet at the following times (in

seconds, counting from the time of cutting the first screen: 0, 0·0666, 0·1343, 0·2031, 0·2729, 0·3439, 0·4159. Find the average velocity between every two successive screens.

Ans. 2,252, 2,216, 2,180, 2,149, 2,113, and 2,083 feet per second.

(9). A body has passed through the space s feet measured from some zero point in its path at the time t seconds, measured from some zero of time; the law of the motion is

$$s = 12{\cdot}2 - 3{\cdot}4\,t + 6{\cdot}7\,t^2.$$

Calculate s when t is 4, calculate s, one tenth of a second later, when t is 4·1. Now find the distance passed through in this tenth of a second, and so find the average velocity.

Thus s is 105·8 when t is 4,
s is 110·887 when t is 4·1.

[The student will notice that when working with a law supposed to be exact we may use many significant figures.]

The space 5·087 feet being passed in ·1 second, there is an average velocity during this tenth of a second

$$= \frac{\text{space}}{\text{time}} = \frac{5{\cdot}087}{0{\cdot}1} = 50{\cdot}87 \text{ feet per second.}$$

Now let the student calculate s for $t = 4{\cdot}01$, and for $t = 4{\cdot}001$, and find the average velocity for shorter and shorter intervals after $t = 4$ and so see what is the actual velocity at $t = 4$.

If s and t are plotted on squared paper (*see* Art. 79), the slope of the curve is a measure of the velocity.

LECTURE V.

Squared Paper—continued

75. When as the result of an investigation we have arrived at a formula connecting y and x, we sometimes try to replace it by an approximate simpler formula.

I have given a good example of this in Art. 14, which students ought now to try with squared paper.

Ex. Plot

$$y = 9{\cdot}9 + {\cdot}6167\,x + {\cdot}35\,x^2 - \frac{1}{15}\,x^3 \quad - \quad - \quad - \quad (1).$$

Find the nearest approximate linear law between the values $x = 0{\cdot}5$ and $x = 3{\cdot}0$.

Ans. $y = 9{\cdot}7 + 1{\cdot}14\,x$ - - - (2).

Of course the student sees that he takes values of x, in each case calculates y, and plots (1) on squared paper. He then finds the straight line which lies most evenly among his points and so finds (2).

Ex. In my theory of Struts it simplified the reasoning enormously—in fact without this, reasoning was practically impossible—to be able to use

$$\frac{a}{1 - bx} \quad \text{and} \quad \frac{1}{\cos\sqrt{x}}$$

as if they were the same. Using values of x from 0 to 0·9, find what values of a and b will be most suitable. Calculate then the value of each of the above functions for each value of x and exhibit in a table.

Ex. In Art. 21 we found that if n is the number of years in which a sum of money will double itself at compound interest at r per cent. per annum, then

$$n = \text{log. } 2 \div \text{log. } \left(\frac{1 + r}{100}\right).$$

Take various values of r, say 2, 2½, 3, 3½, 4, 4½, 5, and in each case calculate n. Plot on squared paper. A curve would pass through the points.

Now to find if there is not a simple approximate law connecting n and r, try various systems of plotting, such as we have tried in other cases. If you plot n and $\frac{1}{r}$, you will find your points lying so nearly in a straight line that you know you may use the approximate formula

$$n = \frac{70}{r}$$

if r is not greater than 5.

Ex. Plot $y = 1 + \sqrt{1 + x^2}$,

Within what limits of values of x may we take

$$y = 1{\cdot}23 + 0{\cdot}98x$$

as equivalent to it. Answer, from $x = 2$ to $x = 10$ or more, Hence we may take $a + \sqrt{a^2 + x^2}$ as nearly equivalent to $1{\cdot}23a + 0{\cdot}98x$ between the limits $x = 2a$ and $x = 10a$.

76. Any equation, however complicated, may be solved.

Example 1. Find a value of x which will satisfy the equation

$$4{\cdot}3\,x^{2{\cdot}56} - 5{\cdot}4\,e^{-0{\cdot}3x} + 17\,x^{-3{\cdot}4} - 12 = 0.$$

To solve this equation, let us say that—

$$y = 4{\cdot}3\,x^{2{\cdot}56} - 5{\cdot}4\,e^{-0{\cdot}3x} + 17\,x^{-3{\cdot}4} - 12.$$

Take various values of x, calculate the corresponding values of y, and plot on squared paper. It is easy to obtain from this an approximation to the value of x which will give $y = 0$. This leads me to take $x = 1{\cdot}2$, and I find $y = 0{\cdot}280$. Now I try $x = 1{\cdot}25$, and I find $y = -0{\cdot}140$. Hence the correct value of x is between these. Plotting these points to a large scale and joining by a straight line, I see that $x = 1{\cdot}23$ is probably right, and on trial I find it sufficiently correct for all practical purposes. By trying 1·231 and 1·229, and plotting the results to a still larger scale, I can obtain even a closer approximation; and, indeed, there is no limit to the accuracy with which one may solve such an equation.

If there are several values of x which will cause y to be 0, of course they also are answers. Every value of x which satisfies an equation is called a *root* of the equation.

Exercise. The following equation may be very easily solved in another way, but the student ought to solve it as an illustration of our squared-paper method of working.

$$x^2 - 5{\cdot}11\,x + 5{\cdot}709 = 0.$$

Let $y = x^2 - 5{\cdot}11\,x + 5{\cdot}709$.

Taking the following values of x, I have calculated the values of y.

x	0	1	1·5	2·0	2·5	3	3·5	4	5
y	5·709	1·599	1·5715	−0·511		−0·621		1·269	5·159

Plotting these on squared paper, I get a curve like fig. 23.

Notice that it cuts OX in two points, P and Q, so that there are two roots to the equation, OP and OQ.

By approximation, as in the last case, near $x = 1{\cdot}64$ or 1·66, and $x = 3{\cdot}45$ or 3·47, I find that the two roots are 1·65 and 3·46.

How many advanced students can solve even a cubic equation? In practical work we may have to solve any equation whatsoever, and we now have a method of doing this with any amount of accuracy that may be desired.

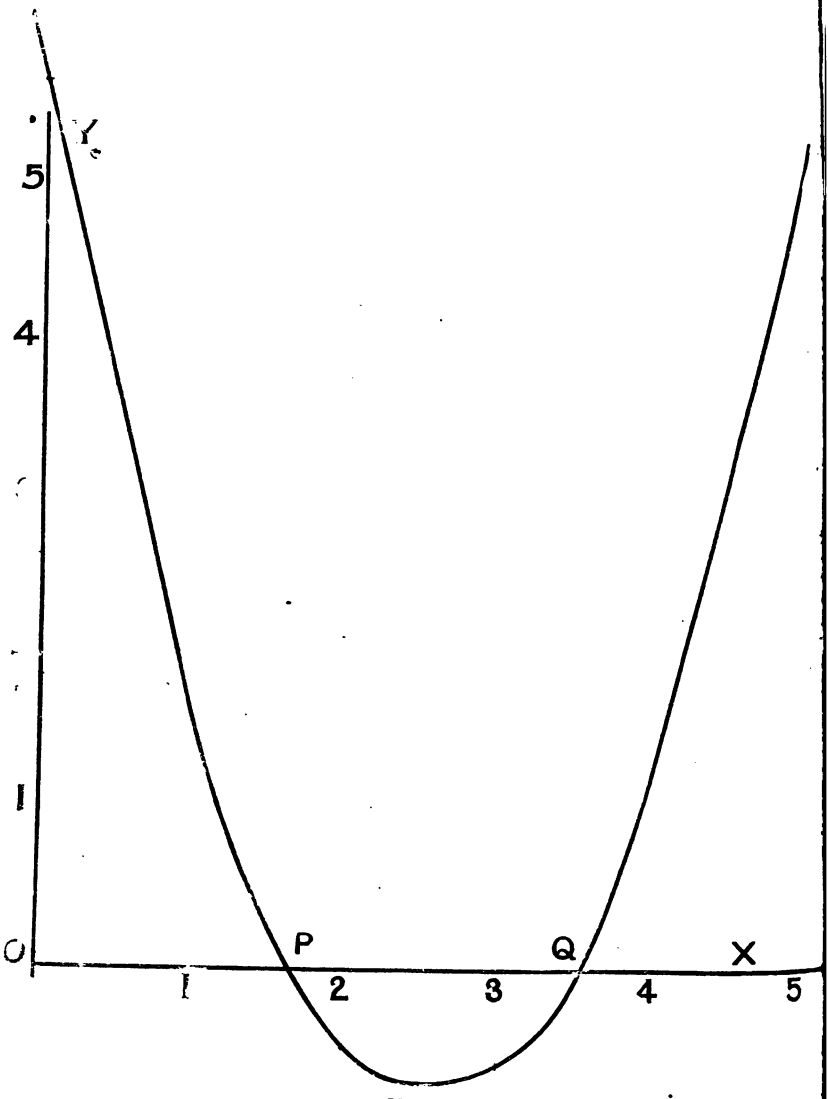


FIG. 23.

Exercise. Find a value of x to satisfy

$$5{\cdot}3\, e^{1{\cdot}04x} \sin^2 0{\cdot}8\, x + 0{\cdot}78\, x^{1{\cdot}52} \cos x - 2{\cdot}126 = 0.$$

The student must remember that $0{\cdot}8\, x$ is in radians, and must be multiplied by 57·296 to convert it to degrees.

Ans. $x = 0{\cdot}74$.

Example. Find the value of x for which

$$\tan x = 2{\cdot}46\, x$$

Let $$y = \tan x - 2{\cdot}46\, x$$

Taking a few values of x at random, and appealing occasionally to squared paper, one is able to get an approximation quickly. Of course, an angle x radians is $57{\cdot}296\, x$ degrees.

x	y
1	− ·903
1·5	+ large
1·3	·4038
1·2	− ·381
1·25	− ·066
1·26	+ ·015

The horizontal lines show the places during the calculation at which I appealed to squared paper to assist me in my next guess.

Ans. $x = 1{\cdot}258$.

77. *Other Examples.*—(1.) Divide the number 10 into two parts such that the product is a maximum.

Let x be one part, then $10 - x$ is the other.

Let the product be called y, or

$$y = x\,(10 - x), \text{ or } 10\,x - x^2.$$

Take various values of x, calculate y in each case, and plot on squared paper. The maximum value of y is evidently given by $x = 5$.

(2.) When is the sum of a number and its reciprocal a minimum? Let x be the number, when is

$$y = x + \frac{1}{x}$$

a minimum? Taking the following values of x we find y in each case:—

x	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{3}$	2
y	$2\frac{1}{2}$	$2\frac{1}{12}$	2	$2\frac{1}{12}$	$2\frac{1}{2}$

On plotting we see that $x = 1$ is the answer.

(3.) What is the greatest volume of a cylindric parcel to be sent by parcel post? The Post Office regulation is that length plus girth must not be greater than 6 feet. Let

x be the radius of the circular section, then $6 - 2\pi x$ is the length. The volume is

$$v = \pi x^2 (6 - 2\pi x).$$

What value of x will make v a maximum? Taking $x = \cdot4$, $\cdot5$, $\cdot6$, $\cdot7$, $\cdot8$, &c., and in each case calculating v, plotting on squared paper we find that $x = 0{\cdot}636$ feet.

(4.) Divide 10 into two parts such that three times the square of one part plus four times the square of the other shall be a minimum.

If x is one part, $10 - x$ is the other. When is

$$y = 3x^2 + 4(10 - x)^2$$

a minimum? Calculating and plotting we find $x = 5{\cdot}71$, and the other part $4{\cdot}29$.

78. Slope of a line.—In Art. 56, $y = a + bx$.

I have called b the slope of the line. We saw that if x increases by the amount 1, y increases by the amount b.

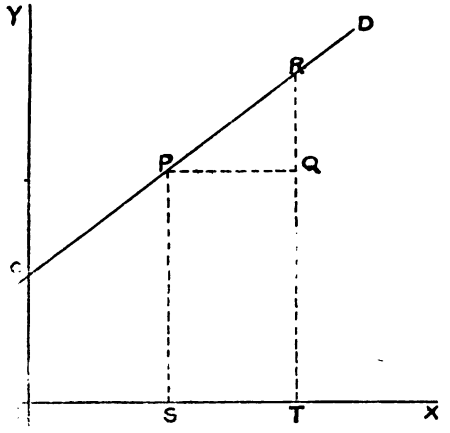


Fig 24.

Let CD be the line; the distance OC represents a. If P and R are two points in the line, and if PQ is 1 then RQ is b.

b is the rise for 1 horizontal.

Note that when we say that a road rises $\frac{1}{20}$ or 1 in 20, we mean 1 foot rise for 20 feet *along* the sloping road. Thus $\frac{1}{20}$ is the sine of the angle of inclination of the road to the horizontal. Our *slope* is different, being the *tangent* of the angle RPQ. [For the meanings of such terms as sine and tangent the student is referred to Art. 100.] Looking upon y as a quantity whose value depends on that of x,

observe that *the rate of increase of y relatively to the increase of x is constant*, being indeed b, the slope of the line. The symbol used for this rate is $\frac{dy}{dx}$. Note that it is one symbol; it does not mean $\frac{d \times y}{d \times x}$. Try to recollect the statement that if $y = a + bx$, then $\frac{dy}{dx} = b$, and that if $\frac{dy}{dx} = b$, then it follows that $y = A + bx$, where A is some constant or other.

Exercise.—If $4x + 3y = 7$, find $\frac{dy}{dx}$;

Here $y = \frac{7}{3} - \frac{4}{3}x$, so that $\frac{dy}{dx} = -\frac{4}{3}$.

Now if a road is of perfectly constant slope it is easy to measure this slope. It rises 1 foot, say, for 10 horizontal, or 2 feet for 20 horizontal, or 3 feet for 30 horizontal. But if the slope of a road is continually altering, and we want to know the slope at a particular place, we really only measure an average slope, however small our distances may be.

79. In the curve of Fig. 25 there is a positive slope (y increases as x increases) in the parts AB, CD, EF, GH, and negative slope (y diminishes as x increases) in the parts DE and FG. The slope is zero at D and F, where y is said to reach *maximum* values, and it is also zero at E and G, where y is said to reach *minimum* values; it is also zero at R, which is neither a point of maximum nor minimum. It must be evident to a student that the slope of a curve at a point is the slope of the tangent to the curve at that point. Suppose we want to know the slope at the point P, Fig. 25. Imagine Fig. 26 to be a great magnification of the curve at point P.

If PD or OI is x and PI is y; let F be another point and let FC or OH be $x + \delta x$ and FH be $y + \delta y$. The student must note that δx is a symbol for a small increment to x; a new kind of symbol. It does not mean a quantity δ multiplied by a quantity x. Then PG is δx and FG is δy, so that $\frac{FG}{PG}$ or $\frac{\delta y}{\delta x}$ is the *average* slope from P to F, and this is really tan FPG.

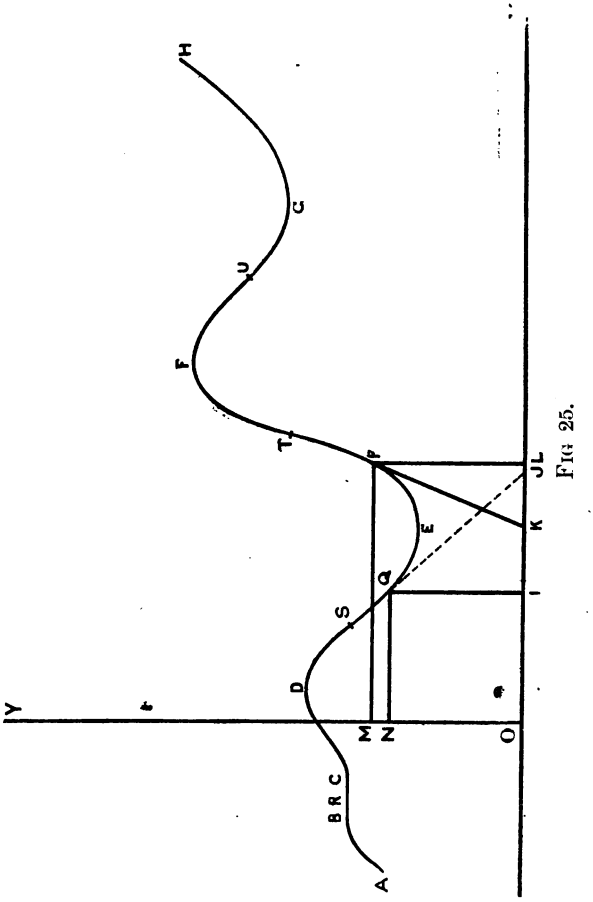


FIG 25.

Now let F be nearer P, say at F′, or nearer still, at F″. In every case the average slope is the tangent of the angle made by FP or F′P or F″P with the horizontal. But it is not until δy and δx are imagined to get smaller and smaller without limit that $\frac{\delta y}{\delta x}$ can be called the actual slope at P. The line joining F and P or F′ and P or F″ and P gets to be more and more nearly what we call the tangent at P as F or F′ or F″ gets nearer and nearer to P.

When we speak of δx getting smaller and smaller and $\frac{\delta y}{\delta x}$ reaching a limiting value we distinguish this

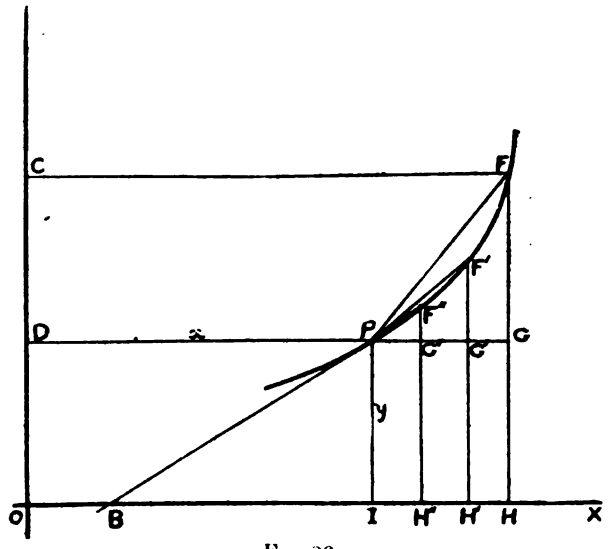


FIG. 26.

limiting value by the symbol $\frac{dy}{dx}$, and it is evidently tan PBX if PB is the tangent to the curve at P. I do not think that anybody can draw a tangent to any curve at a given point very accurately, but if he could he would have no difficulty in finding $\frac{dy}{dx}$ at the point.

In Fig. 25 the slope at P is tan PKX, and is positive. The slope at Q is tan QJX and the tangent of an obtuse angle is negative.

80. If PL = y is, say, **population**, and OL = x means time after some date; the slope at P is $\frac{dy}{dx}$, the rate at which the population is increasing per annum. The scale to which this rate is represented by tan PKL or the slope is easily found, because the rate is

$$\frac{\text{the population represented by PL}}{\text{time represented by KL.}}$$

81. If s is the distance in miles that a train has reached from Euston at the time t, how do we study the motion? If a Bradshaw's railway guide gave, not merely the time of reaching a few stations, but the time of reaching say fifty places between London and Northampton, the record would be something like this, only the student may imagine many more entries; entries perhaps for every half-mile.

	t Time in Hours since leaving Euston.	*s* Distance in Miles from Euston.
10.20 o'clock -	0	0
10.32 „ -	·2	6
10.35 „ -	·25	10
10.40 „ -	·35	14
10.44 „ -	·40	14
10.47 „ -	·45	18
10.53 „ -	·55	24
&c.	&c.	&c.

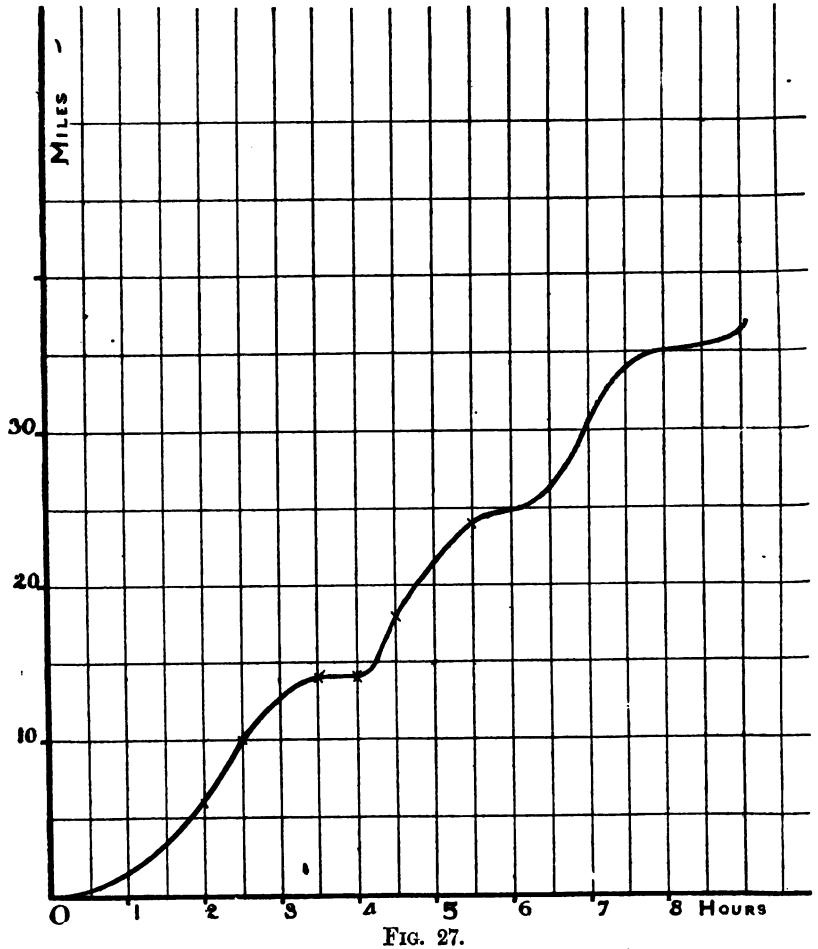


Fig. 27.

If we plot s and t on squared paper as in Fig. 27, it is evident that the slope of this curve everywhere represents the speed of the train. See how the speed quickens on leaving Euston, until from $s = 5$ to $s = 11$ miles it keeps nearly constant, but diminishes and becomes 0 at $s = 14$ miles. In fact, the train has stopped, then we see the speed getting up again, then diminishing and increasing, until at $s = 24$ there is a crawl for a mile.

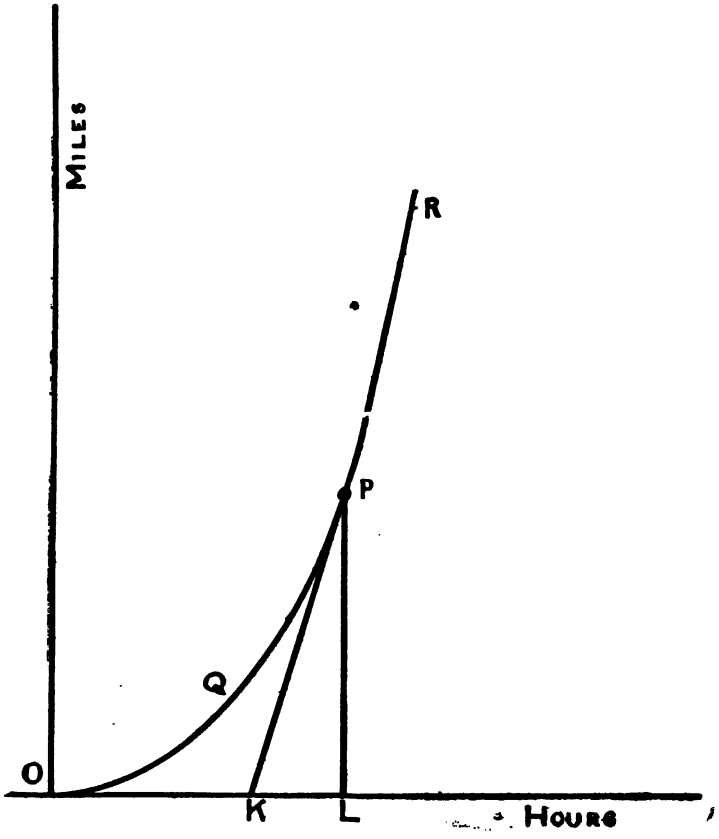
If we represent the train as coming back again, of course, there will be negative slope as the velocity is negative.

Exercise.—The motion of a train leaving Euston is such that $s = 300\ t^2$,

s being in miles from Euston, t in hours from leaving Euston.

At the end of 0·1 hour, find v, the velocity of the train in miles per hour.

Take $t = \cdot 01, \cdot 02$, &c., and in each case calculate s.



IG. 28.

Plot on squared paper. I get the curve O Q P R, Fig. 28. At P, $t = 0{\cdot}1$ hour, $s = 3$ miles; drawing the tangent PK at P, I find that to the scale of time KL represents 0·05 hour and PL is 3 miles. Hence the slope at P represents $\frac{3 \text{ miles}}{\cdot 05 \text{ hour}}$, or 60 miles per hour. As a matter of fact, this is the correct answer, but I did not get it by drawing the tangent. Nobody could draw a tangent so as to get the perfectly correct answer. I got the answer in the following way. I calculated s for the following times:—

δt is the excess time beyond ·1 hour.

δs is the distance in excess of 3 miles.

$\frac{\delta s}{\delta t}$ is the average speed during the interval δt, the space δs being passed over in this interval.

t	s	δt	δs	$\frac{\delta s}{\delta t}$
·1	3			
·11	3·63	·01	·63	63
·101	3·0603	·001	·0603	60·3
·1001	3·006003	·0001	·006003	60·03

It is evident that as we take δt less and less we are getting nearer and nearer to the actual speed at the time ·1.

To be quite sure of this let the work be done algebraically.

At the time t calculate s; again for the new time $t + \delta t$ calculate $s + \delta s$.

We have $s = 300\, t^2$, and $s + \delta s = 300\, (t + \delta t)^2$, or

$$s + \delta s = 300\, \{t^2 + 2\, t\, .\, \delta t + (\delta t)^2\}.$$

Subtracting we get

$$\delta s = 600\, t\, .\, \delta t + 300\, (\delta t)^2$$

Now divide by δt and we get average speed

$$\frac{\delta s}{\delta t} = 600\, t + 300\, .\, \delta t.$$

It is only when δt is made smaller and smaller without limit that we speak of actual velocity $\frac{ds}{dt}$.

Actual velocity at time t is $\frac{ds}{dt} = 600\, t$.

This formula enables us to find the velocity at any time. When $t = 1$ it is 60 miles per hour.

82. The student ought to give a good deal of thought to this idea of a limiting value of $\frac{\delta s}{\delta t}$ as δt is made smaller and smaller.

Of course $\frac{0}{0}$ may have any value we please.

$$\frac{0}{0} = 100 \text{ because } 0 = 100 \times 0.$$

$$\frac{0}{0} = 10{,}000 \text{ because } 0 = 10{,}000 \times 0.$$

Although the student is perfectly familiar with the idea, he had better spend time over it, because although it is so very simple, seemingly, yet it is supposed by mathematicians to be an idea which cannot be communicated to anybody until he has done a great deal of mathematics.

83. Let me give it another form. Suppose y and x to be any quantities whatsoever, connected by the law

$$y = ax^2 \quad - \quad - \quad - \quad (1).$$

Take a particular value of x and calculate y. Now take a new value of x, $x + \delta x$, and calculate the new y, calling it $y + \delta y$

$$y + \delta y = a(x + \delta x)^2,$$

or $$y + \delta y = a\{x^2 + 2x \,.\, \delta x + (\delta x)^2\} \quad - \quad - \quad - \quad (2).$$

Subtracting (1) from (2) we have

$$\delta y = 2ax \,.\, \delta x + a \,.\, (\delta x)^2,$$

divide by δx and we have

$$\frac{\delta y}{\delta x} = 2ax + a \,.\, \delta x \quad - \quad - \quad - \quad (3)$$

Now (3) is true for any value of δx. It gives the *average* rate of increase of y in regard to x. But it is only when we let δx get smaller and smaller without limit that we say

$$\frac{dy}{dx} = 2ax \quad - \quad - \quad - \quad (4)$$

and the actual rate of increase of y with regard to x is known for every value of x and y.

84. Similarly, if $y = ax^3 \quad - \quad - \quad - \quad (1)$

$$y + \delta y = a(x + \delta x)^3$$

$$y + \delta y = a\{x^3 + 3x^2 \,.\, \delta x + 3x \,.\, (\delta x)^2 + (\delta x)^3\} \quad - \quad - \quad (2).$$

Subtracting (1) from (2), and dividing by δx, we get

$$\frac{\delta y}{\delta x} = 3ax^2 + 3ax \,.\, \delta x + a\,(\delta x)^2 \,.$$

Now, letting δx get smaller and smaller without limit, we use the new symbol

$$\frac{dy}{dx} = 3ax^2.$$

85. With a little more knowledge of algebra than I presume in my hearers it will be found easy to prove that if

$$y = ax^n$$

where a and n are any numbers whatever, positive or negative,*

$$\frac{dy}{dx} = nax^{n-1}.$$

This rule comprehends the others that I have given. In fact, if

$$y = a + bx + cx^2 + \&c. + mx^n \quad - \quad - \quad - \quad (1)$$

$$\frac{dy}{dx} = 0 + b + 2cx + \&c. + nmx_{n-1} \quad - \quad - \quad (2)$$

86. In some of the above exercises we had y given as a function of x, and we were asked to find $\frac{dy}{dx}$. We see that this rule is very much easier to apply than any rule that requires us to plot the curve on squared paper. $\frac{dy}{dx}$ is sometimes called the slope of the curve, or the tangent of the angle which the tangent to the curve makes with the axis of x, or it is sometimes called "the **Differential** Coefficient of y with regard to x," or "the rate of increase of y with regard to x."

87. Sometimes we are asked to **Integrate**; that is, when given $\frac{dy}{dx}$ we are asked to find y.

Thus given (2), above, we know that

$$y = \mathrm{C} + bx + cx^2 + \&c. + mx^n$$

The integral of x^3 is $\frac{x^4}{4} + \mathrm{C}$.

„ of x^2 is $\frac{x^3}{3} + \mathrm{C}$.

* The rule fails when $n = -1$. The advanced student knows that if $y = ax^{-1}$, $\frac{dy}{dx} = a \log_e x$.

In every case we add a constant, because the differential coefficient of any constant is 0. Generally we may say that

$$\text{The integral of } ax^m \text{ is } \frac{ax^{m+1}}{m+1}.$$

88. *Exercises.* (1) Differentiate

$$y = 4 + 3x + 0{\cdot}7x^2 + 2{\cdot}15x^3 + 15x^{20} + 12x^{-1} + 24x^{-3{\cdot}04} + 2x^{0{\cdot}786}.$$

Ans.

$$\frac{dy}{dx} = 0 + 3 + 1{\cdot}4x + 6{\cdot}45x^2 + 300x^{19} - 12x^{-2} - 72{\cdot}96x^{-4{\cdot}04} + 1{\cdot}572x^{-0{\cdot}214}.$$

(2) Integrate $3{\cdot}056 + 2x + 15x^2 + 12x^{15} + 1{\cdot}5x^{-2} + 17x^{-{\cdot}046} + 2x^{4{\cdot}565}$.

Ans. $$C + 3{\cdot}056x + x^2 + 5x^3 + \frac{3}{4}x^{16} - 1{\cdot}5x^{-1} + 17{\cdot}8x^{{\cdot}954} + {\cdot}3593x^{5{\cdot}567}.$$

In this answer C may be any constant whatsoever. The student is fairly well aware of the value of a rule for finding $\frac{dy}{dx}$. I will now show the usefulness of Integration.

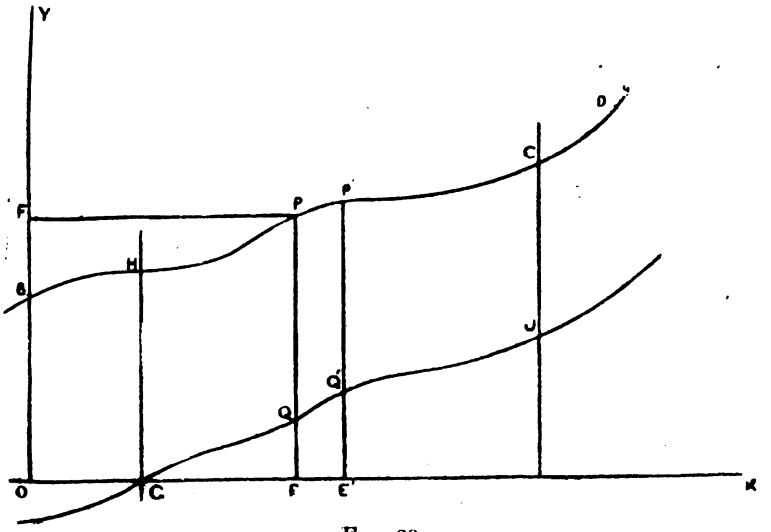


FIG. 29.

Let BPCD be any curve; at the point P let FP or OE be called x, and PE be called y.

Let the ordinate QE of the curve GQJ represent A, the area of HPEG. I have taken HG anywhere as my starting ordinate from which the area is to be counted. Let OE′ be called $x + \delta x$, so that $EE' = \delta x$.

Now $QE = A$, represents area HPEG,

$Q'E' = A, + \delta A$ represents area HP′E′G.

Hence δA is the area PP′E′E.

But as δx is smaller and smaller we find it to be more and more true that

$$\delta A = PE \times \delta x = y \,.\, \delta x.$$

Hence $$\frac{dA}{dx} = y \quad \text{- - - (1).}$$

That is, the ordinate of a curve is the differential coefficient of its area, or area A of a curve is the integral of y, the ordinate of the curve.

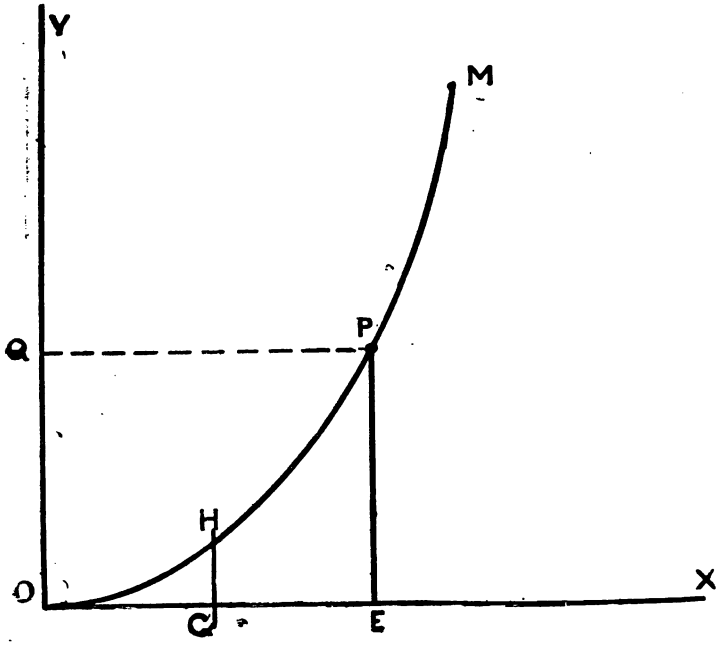


FIG. 30.

89. *Exercise.*—Let OPM be the curve $y = ax^2$.

Then $$A = \frac{a}{3}x^3 + C,$$

where C is some unknown constant which depends upon where we count area from.

Thus if we count area from $x = 0$, that is let $A = 0$ when $x = 0$, then $C = 0$, so that $A = \frac{a}{3}x^3$. Here we have a formula which gives us the area of the curve up to any value of x.

Again. Find the area between the ordinates HG and PE.

A up to PE from some ordinate $= \frac{a}{3}OE^3 + C$.

A up to HG from the same ordinate $= \frac{a}{3} \cdot OG^3 + C$.

Hence area required $= \frac{a}{3}(OE^3 - OG^3)$.

Suppose we want the above areas in terms of PE, &c., instead of a. As $y = ax^2$, and hence $PE = a \times OE^2$ or $a = PE/OE^2$, we can replace a in the above formulæ. Thus area of $POE = \frac{a}{3}OE^3 = \frac{1}{3}\frac{PE}{OE^2} \cdot 0E^3 = \frac{1}{3}PE.OE = \frac{1}{3}$rd of the area of the rectangle QPEO.

90. When using squared paper we saw that many other things are summed up exactly as areas are calculated. Thus if y is the area of cross section of a body with a straight axis at the distance x from one end; if there is a law connecting y and x; the integral of y with regard to x is the volume. Also the integral of xy divided by the volume gives the x of the centre of gravity.

91. *Exercise.*—Suppose the rate of change of the velocity of a body per second $\frac{dv}{dt}$, which we call the acceleration, is constant being a feet per second per second. The integral of this with regard to t is v, the velocity, and it is

$$v = at + c \quad - \quad - \quad - \quad (1)$$

where c is some constant. If v is v_o when $t = 0$ then c is v_o. But v is $\frac{ds}{dt}$, so integrating again we have

$$s = \frac{1}{2}at^2 + v_o t + s_o \quad - \quad (2),$$

for we add a constant on integrating, and we see that this constant must be s_0, the distance of the body from the zero of measurement when $t = o$.

If we differentiate (2) we get (1); if we differentiate (1) we return to our old statement of acceleration = a.

92. Maxima and Minima.—The student will see now that the problems of Art. 77 are easy to solve exactly.

In (1). Divide a number n into two parts such that the product is a maximum. Let x be one part, then $n-x$, is the other. Let the product be called y, then

$$y = x(n-x) \text{ or } y = nx - x^2.$$

Now at a point of maximum or minimum the slope o the curve or $\frac{dy}{dx}$ is 0.

Here $\frac{dy}{dx} = n - 2x$. If we put this equal to 0,

$$n - 2x = 0,\ 2x = n,\ x = \frac{n}{2}.$$

We get the required answer.

It is true that we cannot in this way distinguish as to whether it is a maximum or a minimum that we have found, but there is no great difficulty in finding this out in other ways.

In (2). When is the sum of a number and its reciprocal a minimum? Let x be the number; when is

$$y = x + x^{-1}$$

a minimum? Ans. When $\frac{dy}{dx} = 0$ or,

$$1 - x^{-2} = 0 \quad \text{or,} \quad x^2 - 1 = 0 \quad \text{or,} \quad x = 1.$$

In (3). The volume of the cylindric parcel is a maximum when $\frac{dv}{dx}$ is 0. Now $v = 6\pi x^2 - 2\pi^2 x^3$,

and $\frac{dv}{dx} = 12\pi x - 6\pi^2 x^2$. Putting this equal to 0 and dividing by $6\pi x$ we get $2 - \pi x = 0$

or, $x = \frac{2}{\pi}$, the answer obtained by the use of squared paper.

In (4). Divide a number n into two parts such that a times the square of one part plus b times the square of the other shall be a minimum. As before

$$y = ax^2 + b(n - x)^2 \quad \text{or,}$$
$$y = ax^2 + bn - 2bnx + bx^2 \quad \text{or,}$$
$$y = (a + b)x^2 + bn^2 - 2bnx.$$

Hence $\frac{dy}{dx} = 2(a + b)x - 2bn.$

Putting this equal to o we get

$$x = \frac{bn}{a + b} \text{ the answer.}$$

Exercise 5.—The strength of a beam of rectangular section of given length, loaded and supported in any particular way, is proportional to the breadth of the section multiplied by the square of the depth. If the diameter of a cylindric tree is a, what is the strongest rectangular beam which may be cut from the tree?

Let x be the breadth BC of the rectangle BCEF, fig 31, BE being a. Then CE is $\sqrt{a^2 - x^2}$. Now the strength depends upon $y = \text{BC} \times \text{CE}^2$, and this is

$$y = x(a^2 - x^2) \text{ or } y = a^2x - x^3.$$

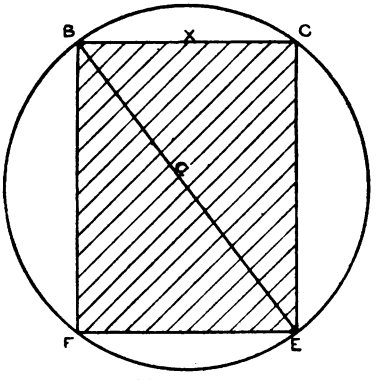


FIG. 31.

When is y a maximum?

$\frac{dy}{dx} = a^2 - 3x^2$. Putting this equal to 0 we find

$$3x^2 = a^2 \text{ or } x = \frac{a}{\sqrt{3}} \text{ or } .5773\,a.$$

93. Of course, differentiation gives us the correct answer, but our old squared-paper method has merits of its own.

For example. Suppose there is a machine consisting of two parts whose weights are x and y. Now suppose the cost of the machine in pounds to be

$$c = 10x + 3y.$$

Also suppose the Power of the machine to be proportional to xy. If the cost is fixed, find x and y so that the Power shall be a maximum.

Here $y = \frac{1}{3}(c - 10x)$ and let $p = xy$ or

$$p = x \times \frac{1}{3}(c - 10x) = \frac{c}{3}x - \frac{10}{3}x^2,$$

$\frac{dp}{dx} = \frac{c}{3} - \frac{20}{3}x$, and if we put this equal to 0 we have $10x = \frac{1}{2}c$. This leads to $3y = \frac{1}{2}c$, that is, the costs of the two parts ought to be equal.

Now suppose it is an actual machine, say a Dynamo, and that x is the weight of the armature part, y the weight of the rest, and that the numbers 10 and 3 were based on real workshop experience. As everything is evidently proportional to c, take $c = 3$, Then

$$p = x - \frac{10}{3}x^2.$$

Now plot x and p on squared paper. It is quite true that we find $x = \cdot 15$ gives the maximum power. But it will be observed that if this exactly best value of x is not employed the power is not much less than the maximum. In fact the squared paper tells us that we do not suffer greatly even by departing considerably from the value of x which is here thought to be the best. Indeed I may say that I employ the differential calculus method to suggest the best answer; but I also use the squared paper method in practical problems.

x	p
·1	·0667
·12	·072
·13	·07367
·14	0·7467
·15	·075
·16	·07467
·17	·07367
·18	·072

94. Instead of saying that the Integral of y with regard to x is area, we use the symbol $\int y.dx$ = area.

$\int$ is meant for a long S and signifies really "the sum of all such terms as." Notice that both $\int$ and dx ought to be used in the symbol for "the integral of y."

The symbol $\int_a^b y.dx$ means "the area of the curve whose ordinate is y from $x = a$ to $x = b$." We already know how to find this. Or, what comes to the same thing, this symbol tells us "Find the general integral of y, insert in it $x = b$, insert in it $x = a$, subtract." This is evident from Art. 89.

LECTURE VI.

Vectors.

95. Hitherto I have only considered scalar quantities, quantities which are dealt with like mere numbers.

A vector quantity, like displacement, velocity, momentum, acceleration, force, impulse, strain, stress, flux of fluid, flux of electric current, magnetic force, magnetic induction, &c., has direction and magnitude. **A vector** can be represented by a straight line; its **magnitude** to some scale by the length of the line; its **clinure** or ort by the clinure of the line; its sense by an arrow-head. Thus the length of the line AB, fig. 32, represents a vector to some scale of measurement; its sense is shown by the arrow-head. Possibly two words are enough, *Size* and *Direction*, if we regard Direction as including both ort and sense.

If A′B′ is drawn parallel to AB and of the same length and sense, it represents exactly the same vector.

96. Addition of Vectors.—If **a**, **b**, and **c**, fig. 33, represent three vectors, To add them; make them the sides of a polygon, fig 34; take care that their arrow-heads are circuital [follow my neighbour]. Thus AB is the same as **a**, CD is **b**, EF is **c**.

Then the last side of the polygon with a non-circuital arrow-head represents their sum. Or, as we write it,

$$\text{AB} + \text{CD} + \text{EF} = \text{AF};$$
$$\text{or,} \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \text{AF} \quad - \quad - \quad (1)$$

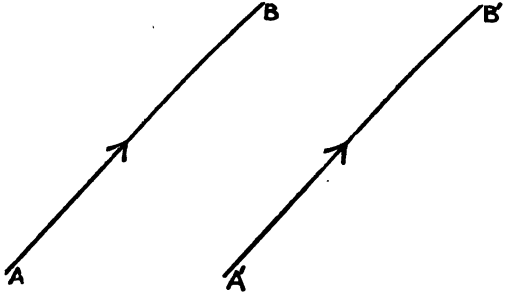
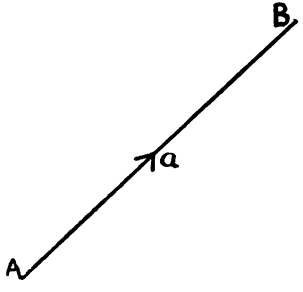


FIG. 32.



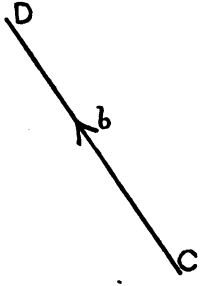


FIG. 33

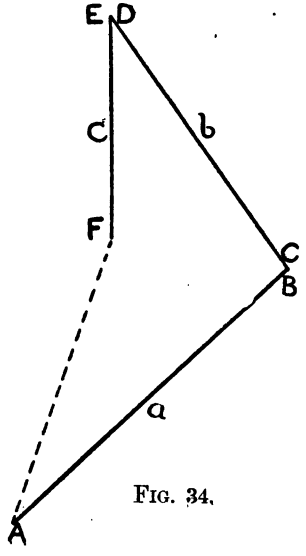


FIG. 34.

The student ought to take three or four vectors and add them according to the above rule, taking them in quite different order as sides of a polygon; in every case the same answer is obtained. In fact $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. We apply now the same symbols as we use in algebra, that is in dealing with scalar quantities, and say that (1) is the same as

$$\mathbf{a} + \mathbf{b} + \mathbf{c} - \mathrm{AF} = o \quad \cdot - \quad - \quad (2)$$
$$\mathbf{a} + \mathbf{b} = \mathrm{AF} - \mathbf{c} \quad - \quad - \quad - \quad (3)$$

Notice that when a letter is used for a vector we employ Clarendon type. Notice that our vectors are not necessarily all in one plane, but when they have all sorts of clinures the polygon is a gauche polygon, and must be illustrated by bits of wire.

We may say that all the above statements make up our definition of a vector.

Vector quantities are such quantities as may be added and subtracted according to the above rule.

It is a self-evident truth that a displacement is a vector If we say that a point is moved to the east 1 foot and to the north-east 2 feet, we draw AB and BC to represent these

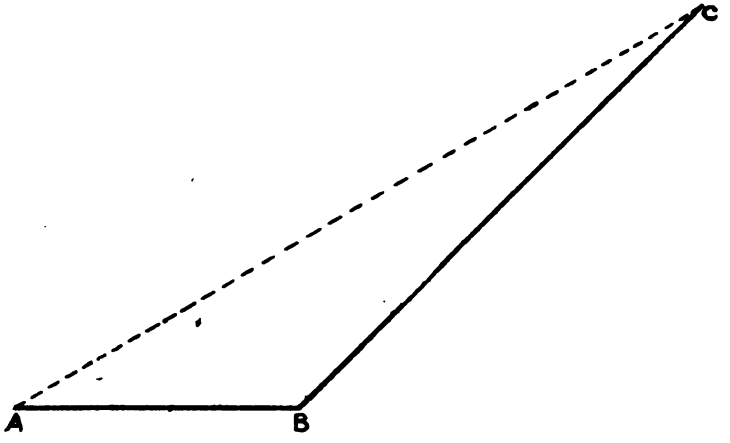


FIG. 35.

two displacements, and we see, fig. 35, that the vector sum of the two, AC, gives the real sum of the two displacements

Here AB + BC = AC,
or AB = AC − BC,
or BC = AC − AB,

so that subtraction of vectors is as easy to understand as addition of vectors.

It is evident that the sum of the displacements AB, BC CD, DE, and EF, Fig. 36, is exactly the same as that of AG GH, and HF; and AF expresses it.

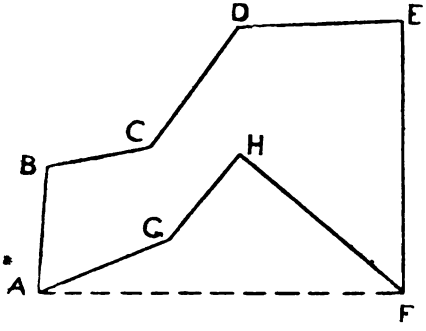


FIG. 36.

Similarly a velocity, being displacement in unit time, is a vector. We look upon the statement that all the above quantities are vectors as a self-evident truth.* There is nothing else to be done. We can no more prove the truth of such a statement than the truth of our own existence. *As soon as we comprehend* the statement we see that it requires no proof. In the same way, acceleration being velocity added per second, is also a vector. Force is mass multiplied upon acceleration, and is therefore a vector also. In so far as a force has an actual position as well as mere clinure, it has a property in addition to that of a mere vector.

Notice that when we use a letter **a** to represent a vector, it is in Clarendon type, and when we see Clarendon type in an equation we know that it is a vector equation without special telling. Also $-\mathbf{a}$ is just the same as **a**, except that its sense, its arrow-headedness, is reversed.

When I was young I spent much time over Duchaylas' proof, that forces are added in the above way. The old 15 page trouble of our youth has disappeared from the books, I am happy to say. It was not only illogical, but stupid, and everybody now recognises this fact. When will the remaining so called "proofs by abstract reasoning" of the schoolbooks disappear? Unfortunately they have a specious appearance of being useful in mind-training, and the stupid teachers can teach nothing else, and so Euclid and its trailing cloud of miseries can disappear only gradually.

* This statement is too sweeping. Advanced students will find that things not vectors may be mistaken for vectors. Thus a *finite* rotation is *not* a vector. The real test comes through Differentiation.

97. It is very difficult to set questions on this subject, because it is so difficult to state clinure or direction. If I use the points of the compass, only a few students will understand. I am afraid that I must ask students to remember the following way. Let OX, fig. 37, be a given standard direction; say, towards the east from O. Now if I always assume the sense of a vector OP to be out from O, and if I state the angle XOP as traced out *anti* clockwise, will the student be able to understand?

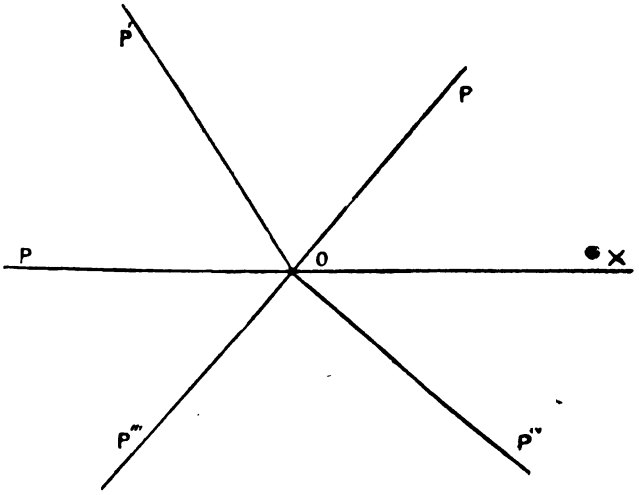


FIG. 37.

Thus, as drawn in fig. 37, I take XOP to be 50°; XOP^{I}, 120°; XOP^{II}, 180°; XOP^{III}, 230°; XOP^{IV}, 320°. That is, I never measure angles in the clockwise direction.

It will now be easy to set questions, I shall state the magnitude of each vector; assume that its sense is out from O, and I shall state the angle it makes with OX.

If $\mathbf{A} = 20, 140°$; $\mathbf{B} = 15, 30°$; $\mathbf{C} = 30, 280°$; $\mathbf{D} = 12, 330°$; $\mathbf{E} = 25, 47°$.

The answers must be stated in the same way. Find by construction:—

(1). $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} + \mathbf{E}$. Answer 30·5 5°·5.

(2). $\mathbf{A} + \mathbf{B} + \mathbf{C} - \mathbf{D} - \mathbf{E}$. Answer 32·6 220°·8.

(3). $-\mathbf{A} - \mathbf{B} - \mathbf{C} + \mathbf{D} + \mathbf{E}$. Answer 32·8 40°·9.

(4). $-\mathbf{A} - \mathbf{B} - \mathbf{C} - \mathbf{D} - \mathbf{E}$. Answer 30·5 185°·5.

As the letter **A** represents a vector, there is much meaning wrapped up in one letter; clinure and sense and the scalar

part which is mere magnitude. If we say $\mathbf{Q} = 1\frac{1}{2}\,\mathbf{P}$, it can only be true if $\mathbf{Q}$ and $\mathbf{P}$ are of the same clinure (or *ort* some people call it) and sense.

It is very well worth while to express a vector $\mathbf{P}$ as $P_o\,\mathbf{p}$ where P_o is the scalar magnitude or *tensor* as it is sometimes called, and $\mathbf{p}$ is said to be the unit vector.

9ℰ If a vector $\mathbf{a}$ changes to $\mathbf{b}$ in the time δt, we see that the vector AB has been added in this time, and we may say

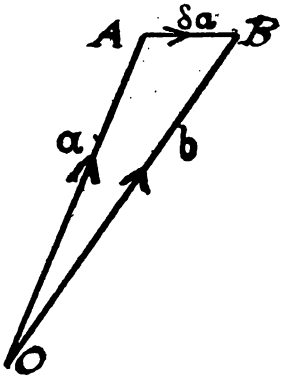


FIG. 38.

that there has been an average increase at the rate $AB \div \delta t$ per second during this time.

We here say $\mathbf{a} + AB = \mathbf{b}$,

or $\mathbf{a} + \delta\mathbf{a} = \mathbf{b}$,

and $\frac{\delta\mathbf{a}}{\delta t}$ is the average rate of increase. This is a vector in the direction AB.

But if we want to know the actual rate of increase at any instant, we must take an interval of time δt which is smaller and smaller without limit. In Art. 81 we considered only rate of increase of mere speed, which is the scalar part of velocity; we must consider rate of change of velocity not only in magnitude but direction, the complete rate of change, if we want to know the real acceleration and consider real force. But the subject of differentiation of vectors is outside the scope of these lectures. When the vector does not change in magnitude, but only in direction, we need only consider angular velocity, *see* Art. 111.

Multiplication of Vector Quantities.—If I were asked to multiply 2 tables by 3 chairs I would not refuse; I would say, 6 chair-tables. But if I were asked to say what

I mean by a chair-table, I would refuse to answer; because nobody has ever given a meaning to the term. But I do know that when this sort of thing comes into a Physical Problem we can always give a useful meaning. This is beyond ordinary Algebra, and yet our processes are carried on by the rules of Algebra. Observe that I do not say that

$$2 \text{ tables} \times 3 \text{ chairs} = 3 \text{ chairs} \times 2 \text{ tables}.$$

Whether this is or is not true depends upon the meaning we attach to the whole process. It is easy to multiply any quantity whatever by a mere number. For example, 2 tables $\times$ 3 = 6 tables. Also a velocity of 6 feet per second due east multiplied by 3 is 18 feet per second due east. Again, if we multiply a vector by any scalar quantity we get a vector in the same direction, although the meaning, the name of its unit, may not yet have been fixed. A velocity of 6 centimetres per second due east multiplied by 3 grammes is a momentum of 18 gramme-centimetres per second due east. Thus, then, the multiplication of a vector by any scalar quantity is not difficult to understand; the result is a vector in the same direction. But what meaning are we to attach to the multiplication of vectors by one another?

99. Surely you are all acquainted with examples. If a vertical force of 4 lb. acts through a vertical distance of 3 feet, we say that it does the work

$$3 \text{ ft.} \times 4 \text{ lb.} = 12 \text{ foot-pounds}.$$

Now these are two vectors that I have multiplied together. In fact, the product of a force into a displacement is quite familiar to us. If there is any force $\mathbf{A}$, and if the body that it acts upon has a displacement $\mathbf{B}$ we say that the work done is $\mathbf{AB}$.

Some people write this $S \cdot \mathbf{AB}$ because work done, energy, is a scalar quantity which has no direction and is not a vector, but I prefer to leave the S out. Now notice that we use $\mathbf{AB}$ to represent the work done, whatever may be the clinures and senses of $\mathbf{A}$ and $\mathbf{B}$.

It is well worth while to study this example carefully, for this product of two vectors exactly illustrates what we mean by the scalar product of any two vectors.

Note that my unit of force is that used by the engineer, the weight of one pound at London.

Let us suppose that the force $\mathbf{A}$ is A lb., but it is evidently something more; I must state the direction. Let it be vertically upwards. If a force of 1 lb. acting vertically upwards be completely indicated by the letter $\mathbf{a}$, then

$$\mathbf{A} = A\,\mathbf{a}$$

That is **a** is the unit force vector and A is a mere numeric.*

Again, suppose **B** to be a displacement of B feet in some direction which may not be vertically upwards; suppose that the letter **b** represents the vector displacement of

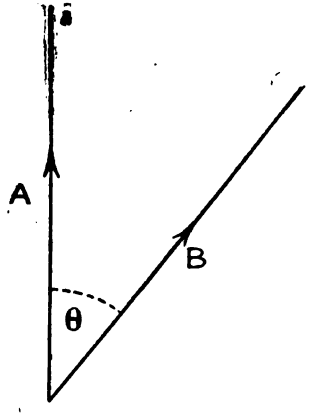


FIG. 39

1 foot in that actual direction of **B**, that is, that the letter **b** signifies direction as well as 1 foot, in fact

$$\mathbf{B} = B\,\mathbf{b}.$$

Then we can say that

$$\mathbf{AB} = \mathrm{AB}.\mathbf{ab},$$

---

* It is of philosophic interest to ask whether it would not be better to put the idea in the following form.

When we speak of a force of A lb. being vertically upwards, observe that this is the result of a mental process which induces us to think of representing the abstract notion of a force by a line drawn vertically upwards with an arrow head upon it. I might speak of a line of the unit length which is used as the scale drawn vertically upwards as the unit displacement or space vector upwards. Suppose I denote it by the letter *i*. Then

$$\mathbf{A} = A\,\mathbf{i}$$

where *A* is not a mere number; it is *A* lb.. and all our unit vectors will be mere unit displacements. When, then, we speak of products of vectors we should always mean the products of unit displacement vectors multiplied by the products of the scalar quantities. For example, in the above case **AB** would really mean (if **s** is unit displacement in the direction of **B**)

$$\mathrm{A\,lb.} \times \mathrm{B\ feet} \times \mathbf{i}\mathbf{s}.$$

In truth the distinction between the two methods of working is only of academic interest. I have no doubt that much printer's ink will be wasted over the question when the subject of vectors becomes part of every school course of study. Mr. Heaviside prefers to consider the unit vector as a mere space displacement, as I do in this note. Thus the product of two unit vectors will really always be the cosine of the angle between their directions.

The product of two mere numbers A and B is easy enough, but what signification are we to give to the product **ab** of two unit vectors; **a**, one pound vertically upwards and **b** a displacement of one foot in some direction not necessarily vertically upwards?

It is exactly our chair-table question. We are asked to give a meaning to the product of our units, **ab**. Now we know that meaning if in the above case we are going to say—

$$\mathbf{AB} = \text{work done.}$$

Because the work done when the force **A** acts through the displacement **B** is obtained by finding the resolved part of **A** in the direction of **B** (we call this A cos θ lb. if θ is the angle between the vectors), and multiplying by B feet, our answer being AB cos θ foot pounds. Or we might have done it by finding the resolved part of **B** in the direction of **A** (this is B cos θ feet) and multiplying by A pounds, our answer as before being AB cos θ foot pounds.

Since then

$$\mathbf{AB} = \text{AB}.\mathbf{ab} = \text{AB cos } \theta \text{ foot pounds,}$$

we see that the product of

a which is a force of 1 lb. vertically upwards,

b which is a displacement of 1 foot making an angle θ with the upward direction,

is cos θ foot pounds. The Student should here note that *ab* is the same as *ba*.

Leaving out of account mere names of unit quantities, we may say then that our definition of the meaning of the product of two unit vectors is

$$\cos \theta.$$

where θ is the angle between their directions, and the meaning of the product **AB** is AB cos θ where A and B are the *tensors* or scalar magnitudes, and θ is the angle between the directions of the vectors. To measure θ, always draw the two vectors from a point 0 with their arrow-heads going out from 0. As we deal only with a cosine of this angle, it will be found that it may be measured either clockwise or anti-clockwise.

Exercises.—(1). A force of 350 lb. acts on a tram-car in a direction towards 20° west of north; the velocity of the car is 20 feet per second due north (or $\theta = 20°$). Find the Power being given to the car. (Generally when we multiply (vectorially) any kind of force by any kind of velocity we are in the habit of calling the answer "the activity.")—Answer, 6,578 foot pounds per second.

(2). In the above case the force acts towards due west (or $\theta = 90°$). What is the power or activity?—Answer, 0.

(3). In the above case the force acts towards the south-west (or $\theta = 135°$). Find the power.—Answer, minus 4,949 foot pounds per second.

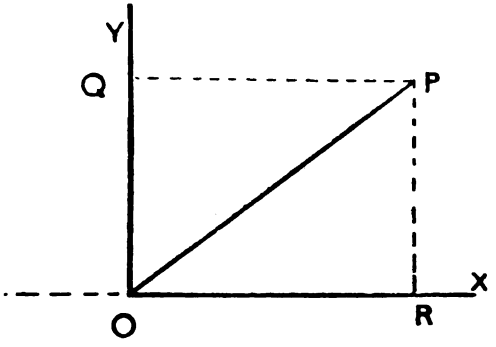


FIG. 40.

(4). In the above case the force acts towards the south (or $\theta = 180°$). Find the power.—Answer, minus 7,000 foot pounds per second.

100. If a student did not know beforehand what cos θ is, he ought to see what it is from the above description.

Let there be a unit vector OP making an angle θ with the vector OX. Project OP upon OX, and the length of the projection OR is what we call *cos* θ.

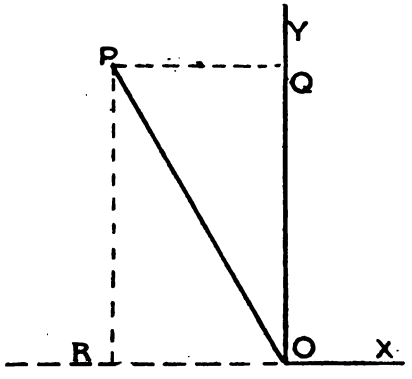


FIG. 41.

Or indeed, if OP is of any length, the length of OR divided by the length of OP is cos θ.

Again let OY be another standard direction at right angles to OX. Project OP upon OY and we have OQ as the resolved part of OP in the direction OY, just as OR was its resolved part in the direction OX. Hence OQ ÷ OP =

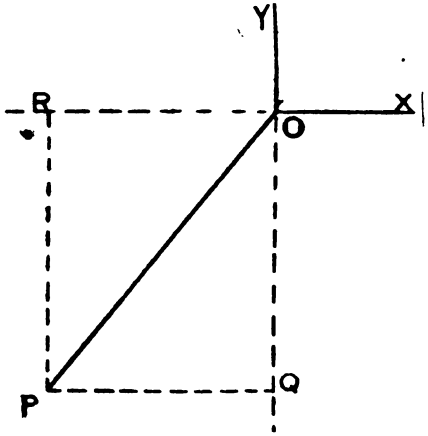


FIG. 42.

cos YOP. Now the length of OQ or RP divided by OP is also called the sine of ROP or sin θ. In fact we have the three definitions

$$\frac{RP}{OP} = \sin\theta, \frac{OR}{OP} = \cos\theta, \frac{PR}{OR} = \tan\theta \quad . \quad . \quad . \quad (1).$$

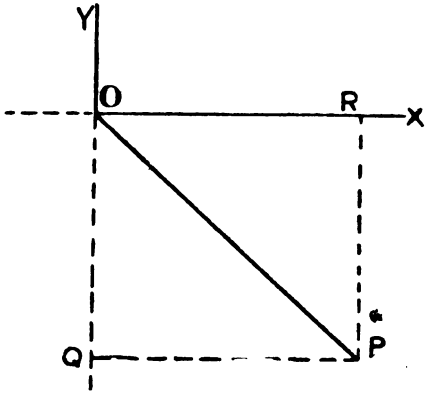


FIG. 43.

Now consider an angle θ greater than 90° and less than 180°. OR, fig. 41, is now negative as compared with the standard direction OX, OQ is positive still. So that when

we make the three statements (1) and make measurements, we find that sin θ is positive, cos θ is negative, tan θ is negative.

For $\theta > 180°$ and $< 270°$, OQ is negative, fig. 42, and OR is negative, so that sin θ is negative cos θ is negative, tan θ is positive.

Similarly for $\theta > 270°$ and $< 360°$, fig. 43, we find that sin θ is negative and cos θ is positive and tan θ is negative.

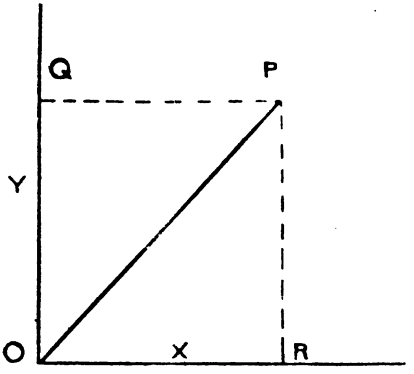


FIG. 44

101. We see that if OX and OY are two standard lines at right angles in a plane, the position of a point P, fig. 44, may be defined in two ways.

(1). If we know x its distance to the right of OY and y its distance above OX.

(2). If we know r its distance from the point O and the angle θ which r makes with OX.

We see that the connection between these co-ordinates is $x = r \cos \theta$, $y = r \sin \theta$, and of course $r^2 = x^2 + y^2$. Also $y/x = \tan \theta$. Given x and y we can find r and θ, or given r and θ we can find x and y.

Exercise.—Draw angles of 35°, 140°, 230°, 310°. Measure off distances OP great enough for fairly accurate work. In each case drop a perpendicular from P upon the other limb of the angle to Q. Measure the distances OQ and QP and notice whether they ought to be considered negative or positive. In every case find the sine, cosine, and tangent. I give the answers in a table.

θ	sin θ	cos θ	tan θ
35°	·5736	·8192	·7002
140°	·6428	–·7660	–·8391
230°	–·7660	–·6428	1·1918
310°	–·7660	·6428	–1·1918

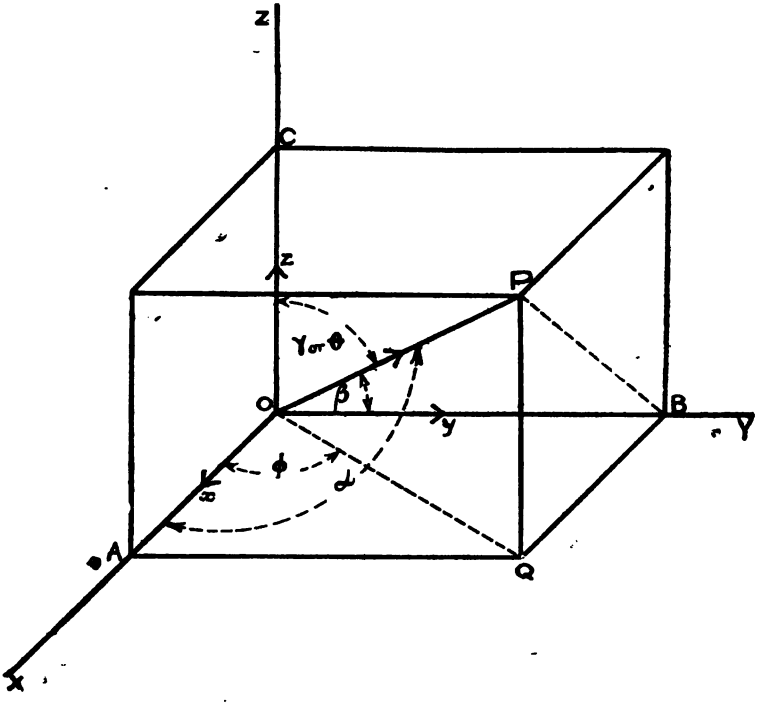


FIG. 45.

The simple principle to keep in one's mind is, that a vector OP projected upon any direction OX is a vector OR whose amount is OR = OP . cos θ if θ is the angle between them. In the above case, notice that OQ = OP . cos QOP. But we defined it as OQ = OP . sin QOP. Hence.

$$\sin QOP = \cos ROP.$$

Indeed when the sum of two angles is a right angle (we call them complementary) the sine of either is the cosine of the other. I need not here dwell further upon these trigonometrical functions, as they are called

102. When we deal with vectors in all sorts of directions in space it is necessary* to use three standard directions, OX, OY, and OZ mutually at right angles. A vector OP is projected as OA in the direction OX. By our old definition, if α, β, γ are the angles which OP makes with OX, OY, OZ, then OA $= x =$ OP $\cdot \cos \alpha$, OB $= y =$ OP $\cdot \cos \beta$, OC $= z =$ OP $\cdot \cos \gamma$. Two of these three angles ought to be given if we want to specify the direction of a vector.

Now it is easy to show that $OP^2 = OA^2 + OB^2 + OC^2$ so that $OP^2 \cdot \cos^2 \alpha + OP^2 \cos^2 \beta + OP^2 \cos^2 \gamma = OP^2$, or, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

103. Hence when a man, in telling me the direction of a vector, gives me its α, β and γ, I have one check on the accuracy of his measurement.

Ex. 1. I have carefully measured and found in a certain case that $\alpha = 20°$, $\beta = 75°$, $\gamma = 76°{\cdot}5$.

Calculate what γ ought to be from the other two and state the percentage error in the measurement of γ, if the others are quite correct.

Ans. 77·08°, percentage error $100 \times \frac{\cdot 58}{77 \cdot 08}$ or 0·75 per cent.

Ex. 2. The following values of OP and of α and β are given for three vectors. Find γ in every case. Find the projections in the three standard directions. Add each set up.

Values of OP.	α	β	γ as calculated.	OA.	OB.	OC.
30	70°	37°	60°·3	10·26	23·96	14·85
25	150°	84°	60°·7	−21·65	2·612	12·23
15	85°	170°	81°·0	1·308	−14·77	2·346
Sums				−10·08	11·80	29·43

Ex. 3. Find the vector whose three projections are the above sums.

* Of course I do not mean that it is necessary. A single vector **a** when given, enables us to specify all vectors in space parallel to **a**. Given another independent vector **b** (or parallel to **b**), we can specify any vector of any size in the plane of **a** and **b** (or in any parallel plane). Thirdly, any third independent vector **c** enables us to specify any vector whatever as $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$.

Ans. $OP = \sqrt{(10{\cdot}08)^2 + (11{\cdot}8)^2 + (29{\cdot}43)^2} = 33{\cdot}27$,
and it makes angles α, β, and γ with OX, OY, and OZ, such that

$$\cos \alpha = -{\cdot}3029, \text{ or } \alpha = 107^\circ{\cdot}7,$$
$$\cos \beta = \ {\cdot}3547, \text{ or } \beta = \ 69^\circ{\cdot}2,$$
$$\cos \gamma = \ {\cdot}8845, \text{ or } \gamma = \ 27^\circ{\cdot}8.$$

We have therefore found the sum of the above three vectors. The student who knows descriptive Geometry will find the sum of the above three vectors by construction and see if he gets the same answer.

The projection of a vector OP upon a plane XOY, Fig. 45, is OQ or OP cos POQ. The projection of OP upon the line OZ is OC or OP cos POZ. Let POZ be called θ, we have seen that POQ is the complement of θ,

$$OC = OP \,.\, \cos\theta, \ OQ = OP \,.\, \sin\theta.$$

Now let the angle QOX be called ϕ; then $OA = OQ \,.\, \cos\phi$ and $OB = OQ \,.\, \sin\phi$. If now we let the length of OP be supposed to be always positive and we call it r; if OA is called x, OB, y; OC, z; these being the projections of r on OX, OY, and OZ, the three mutually rectangular standard directions, we have two ways of stating our results, both of which are in use.

$$x = r\cos\alpha \text{ or } x = r\sin\theta \,.\, \cos\phi;$$
$$y = r\cos\beta \text{ or } y = r\sin\theta \,.\, \sin\phi;$$
$$z = r\cos\gamma \text{ or } z = r\cos\theta.$$

It will be noticed that γ in the one case is the same as θ in the other, and

$$x^2 + y^2 + z^2 = r^2;$$

and also $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

104. If we think of the plane XOY as the equatorial plane of the earth, and OZ as the axis of the earth, then the position of a point P in or on or outside the earth which moves with the earth is given if we know its r, or distance to the centre of the earth, its θ or co-latitude, and its ϕ or east longitude. ϕ is evidently the angle which the plane containing OZ and OP and OQ (called a meridianal plane) makes with the standard meridianal plane ZOX (in our analogy this is the meridian through Greenwich). It is to be remembered, however, that the α, β, γ way is also very greatly used to specify direction. Cos α, cos β and cos γ are called the "direction-cosines" of a line, and the letters l for cos α, m for cos β, n for cos γ are very greatly used. So that $x = lr$, $y = mr$, $z = nr$, and $l^2 + m^2 + n^2 = 1$.

We usually specify the inclination of a plane by stating the direction cosines of its normal, and of the surface of a body at any place by the direction cosines of the normal to the surface at that place.

I see that I have instinctively left the consideration of vectors in general and applied my rules to mere displacement vectors. In fact we have at length reached the subject of Geometry; that is, the relation of mere lines to one another. If we consider XOZ, ZOY, and YOX as three mutually perpendicular standard planes of reference, the position of a point P is fixed if we know its x, y and z; or its r, θ and ϕ; or its r, and any two direction cosines. And if we know the position of a point in any one of these three ways, we can calculate the other dimensions.

105. *Exercises.* (1.) The x and y co-ordinates of a point in a plane are 3 and 4; find r and θ. As $r \cos \theta = 3$ and $r \sin \theta = 4$,

$$\frac{4}{3} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta, \text{ or } \theta = 53°{\cdot}13 \text{ nearly}; \; r^2 = x^2 + y^2 = 25,$$

so that $r = 5$.

(2.) If a point has $r = 10$ inches, $\theta = 37°$ find x and y; $x = 10 \cos 37° = 7{\cdot}986$ inches, $y = 10 \sin 37° = 6{\cdot}018$ inches.

(3.) If $r = 10$ inches, $\theta = 35°$, $\phi = 62°$, find x, y, z;

Here $x = 10 \sin 35° \cos 62° = 2{\cdot}693$ inches.

$y = 10 \sin 35° \sin 62° = 5{\cdot}064$ inches.

$z = 10 \cos 35° = 8{\cdot}192$ inches.

(4.) If $x = 3$, $y = 5$, $z = 6$, find r, θ and ϕ

Now $r^2 = x^2 + y^2 + z^2$, so that $r = 8{\cdot}370$.

$$z = r \cos \theta, \text{ or } \cos \theta = \frac{6}{8{\cdot}37} = {\cdot}7171 \text{ or } \theta = 44°{\cdot}18,$$

$$3 = 8{\cdot}370 \sin 44°{\cdot}18 \times \cos \phi,$$

$$\text{or } \cos \phi = 3 \div (8{\cdot}370 \times {\cdot}6969) = {\cdot}5144,$$

$$\text{or } \phi = 59°{\cdot}03.$$

(5.) If $x = 3$ $y = 5$, $z = 6$, find r, l, m, n.

Answer as in last case, $r = 8{\cdot}370$, $l = x/r$ or ${\cdot}3585$, $m = y/r$ or ${\cdot}5975$, $n = 3/r$ or ${\cdot}7171$.

106. Exercises on Scalar Products.—*Ex.* (1.) The flow of fluid per unit area through a surface is $\mathbf{nV}$ where $\mathbf{n}$ is the unit vector normal to the surface, and $\mathbf{V}$ is the velocity of

the fluid. If the fluid flows at 5 feet per second at an angle making 35° with the normal to the surface, find the flow per square foot. Answer, 5 cos 35° or 4·096 cubic feet per second.

(2.) If **E** is electric force and **D** is electric displacement, it is important to calculate half of **ED**. If **E** = 250 vertically downwards, and if **D** = 1·56 making an angle of 42° with the downward direction, find $\frac{1}{2}$ **ED**.—Answer, $\frac{1}{2} \times 250 \times 1{\cdot}56 \times \cos 42° = 289{\cdot}8$.

(3.) If **A** and **B** are the sides of a parallelogram, **A** + **B** and **A** − **B** are the diagonals.

$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\,\mathbf{AB} + \mathbf{B}^2$. Show that this is really the ordinary formula for the length of a diagonal.

$(\mathbf{A} - \mathbf{B})^2 = \mathbf{A}^2 - 2\,\mathbf{AB} + \mathbf{B}^2$. Show that this is really the ordinary formula for the length of the other diagonal.

Express the geometrical meaning of

$$(\mathbf{A} + \mathbf{B})^2 + (\mathbf{A} - \mathbf{B})^2 = 2\,(\mathbf{A}^2 + \mathbf{B}^2),$$

and $$(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2 = 4\,\mathbf{AB}.$$

(4.) If **A**, **B**, and **C** are the edges of a parallelopiped, show that the ordinary formula for the length of the diagonal is given by $(\mathbf{A} + \mathbf{B} + \mathbf{C})^2 = \mathbf{A}^2 + \mathbf{B}^2 + \mathbf{C}^2 + 2\,\mathbf{AB} + 2\,\mathbf{AC} + 2\,\mathbf{BC}$.

(5.) If a plane figure moves, it traces out a volume (per unit of its area) equal to the scalar product **ND** where **N** is the unit vector normal to the plane area, and **D** is the displacement of the centre of area. If the area keeps parallel to itself (in the final position it may rotate about an axis through the centre of area, and this will make no change in the answer) **N** is same in every position, and **D** will then be the resultant or total displacement.

When I was asked what I meant by

2 tables × 3 chairs = 6 chair-tables

I refused to answer. Algebra had done its duty; it could go no farther. I may now give a meaning to

1 chair-table

and invent a new science. But I may give another meaning and invent another science, and both sciences may be useful. So when we multiply vectors we can give a scalar meaning to our answers as we have already seen. But there is no reason why we should not give another meaning, a vector meaning, and that in certain cases this also should be of value.

107. There is **another way*** in which the **product** of vectors enters into calculations. Suppose we have a magnetic field **B**. That is, it is of the amount B in the direction OA, fig. 46, and amount and direction are both specified by the letter **B**. Let there be a conductor in the field with a current **C** flowing in it. Here again **C** signifies that there is a current of the amount C flowing in the direction OG. We know that there is a mechanical force acting on the conductor in the direction OD, which is at right angles to both OA and OB, and its amount is BC sin θ. If this force is denoted by **F** we say that

$$\mathbf{F} \text{ is the vector Product } \mathbf{C}\,\mathbf{B}$$

$$\text{or,} \qquad \mathbf{F} = \mathrm{V}\cdot\mathbf{C}\,\mathbf{B}$$

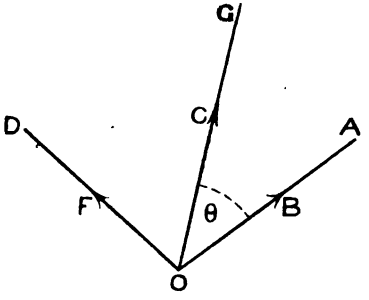


FIG. 46.

Definition.—The **Vector Product** of two vectors **C** and **B** is a vector at right angles to both, its tensor being the product of the tensors of **C** and **B** multiplied by the sine of the angle between them; that is, the area of the parallelogram of which OA and OG are two sides.

Ex.—A body is spinning about an axis OX at A radians per second, and the letter **A** represents both amount of spin and direction OX. In fact A is the tensor of **A**, and **A** has the direction OX. A point P in the body is at the distance r = OP from O, and the letter **R** signifies the amount r or OP and also the direction of OP. It is evident that the velocity **V** of P at any instant is specified both in magnitude and direction by

$$\mathbf{V} = \mathrm{V}\cdot\mathbf{A}\mathbf{R}.$$

* Any combination in products of the components of two vectors might perhaps be called some special kind of product of the vectors; but our two are the only useful ones; furthermore, as they are independent of the axes of co-ordinates, they are the ones that occur in Nature

for the amount of it (its tensor) is Ar. sin XOP, and its direction is at right angles to the plane XOP. It is outside the scope of these elementary lectures to dwell further upon the two kinds of products of vectors which enter so much into physical calculations. I may not even dwell upon the rule for the sign of V **AB** or show that V **AB** = − V **BA**.

108. If I had time I would arrange the next part of my subject better. We have at length arrived at Geometry, which is possibly a little too advanced a subject for an elementary course in Mathematics; but it will be some time before it is considered to be so, and for the present one must assume that students know something of this complex vector subject even before they have done much Algebra. It will be seen that I have assumed that you had some elementary knowledge of Geometry even at the beginning of these

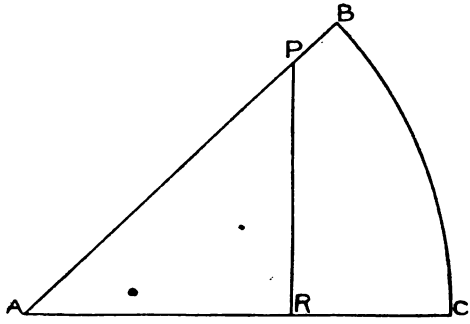


FIG. 47.

lectures. I cannot now do better, I think, than give a few statements about angular measurement, possibly better suited to your comprehension than what I have already given.

109. Angle.—An angle can be drawn: First, if we know its magnitude in *degrees*; a right angle has 90 degrees. Second, if we know its magnitude in *radians*; a right angle contains 1·5708 radians. Two right angles contain 3·1416 radians. One radian is equal to 57·2958 degrees. One radian has an arc BC equal in length to the radius AB or AC. It sometimes gets the clumsy name "a unit of circular measure." Third, we can draw an angle if we know either its *sine*, *cosine*, or *tangent*, &c. Draw any angle BAC (Fig. 47). Take any point, P in AB and draw PR at right angles to AC. Then measure PR, AP, and AR in inches and decimals of an inch

PR ÷ AP is called the *sine* of the angle. AR ÷ AP is called the *cosine* of the angle. PR ÷ AR is called the *tangent* of the angle.

Calculate these for any angle you may draw, and measure with a protractor the number of degrees in the angle. You will find from the Mathematical Tables whether your three answers are exactly the sine, cosine, and tangent of the angle. This exercise will impress on your memory the meaning of the three terms. It will also impress upon us the fact that if we know the angle in degrees, we can find, by means of our tables, its sine, or cosine, or tangent; and if we know any one of the sides AP or PR or AR of the right-angled triangle APR and the angle A, we can find the other sides. Divide the number of degrees in an angle by 57·2958 and we find the number of radians. Suppose we know the number of radians in the angle BAC, and we know the radius AB or AC, then the arc BC is

A B × number of radians in the angle. Given, then, a radius to find the arc, or given an arc to find the radius, are very easy problems.

A student becomes accustomed, on seeing an angle drawn on paper, to judge from a mere glance how many degrees the angle contains. It would be an advantage to acquire the habit of judging how many radians there are in the angle.

What we mean is, that he ought to be as ready to *think* in *radians* as in *degrees*, and to do this he requires to be familiar with the size of a radian.

110. Exercises. (1.) Draw an angle of 35°. Find by measurement the sine, cosine, and tangent of the angle, and compare with the numbers in the tables.

Calculate the number of radians.

Try if $\sin^2 35° + \cos^2 35° = 1$; if $\sin 35° \div \cos 35° = \tan 35°$; if $\tan^2 35° + 1 = \sec^2 35°$ where sec. A means $1 \div \cos A$.

(2.) The sine of an angle is 0·25; find its cosine and tangent. Find the angle by actual drawing. How many radians? *Ans.*, ·9683, ·2582, 14°·5, ·2528.

(3.) What are the sine, tangent, and radians of $1\frac{1}{2}$ degrees? *Ans.*, Each ·0262.

(4.) If in Fig. 47 A is 47°, and AP 5·23 feet, find AR and PR. *Ans.*, AR = 3·567, PR = 3·824.

(5.) The arc BC Fig. 47 is 12 feet, AC is 16 feet; find the angle. Calculate AP and PR if AR is 10·5 feet.
Ans., ·75 radians 42° 58′. AP = 14·32 ft., PR = 9·779 ft.

(6.) The altitude of a tower observed from a point distant 200 yards horizontally from its foot is 19°·3; find its height.
Ans., 70·03 yards.

(7.) A mast consists of two parts, AB and BC. From a point in the horizontal from A, AB subtends 35° and AC subtends 45°; find the ratio of AB to BC. *Ans.*, 2·335.

(8.) From the top of a hill the angles of depression from the horizontal of two consecutive milestones in a line with the hill on a straight level road were found to be 12° and 13°; find the vertical height of the hill above the road.
Ans., 0·2236 mile.

(9.) A point is in latitude 25°. If the earth were a sphere of 3,960 miles radius, how far is the point from the axis? What is the length of the circumference of the circle called a parallel of latitude that passes through the point? What is the 360th part of this in length, and what is it called? What is the 360th part of a circle which is a meridian? What is it called? Answers. Distance from axis, 3,589 miles; circumference, 22,550 miles; $\frac{\text{circum.}}{360}$, 62·63 miles; $\frac{\text{meridian}}{360}$ = 69·10 miles.

111. Angular Velocity.—If a wheel makes 90 turns per minute, this means that it makes 1·5 turns per second. But in making one turn any radial line moves through the angle of 360 degrees, which is 6·2832 radians; so that 1·5 turns per second means 6·2832 × 1·5, or 9·4248 radians per second. This is the common *scientific way* in which the *angular velocity* of a wheel is measured—so many radians per second.

If a wheel makes 30 turns per minute, its angular velocity is 3·1416 radians per second; n turns per minute mean $2\pi n$ radians per minute, or $2\pi n \div 60$ radians per second. One turn is the angular space traversed in one revolution.

Exercise. Show that the linear speed in feet per second of a point in a wheel is equal to the angular velocity of the wheel multiplied by the distance in feet of the point from the axis.

112. Angular Acceleration.—The increase of angular velocity per second. If a wheel starts from rest, and has an angular velocity of 1 radian per second at the end of the first second, its average angular acceleration during this time is 1 radian per second per second.

113. *Exercises.*—(1.) A shaft revolves at 800 revolutions per minute. What is its angular velocity in radians per second? *Ans.*, 83·79.

(2.) A point is 3,000 miles from the earth's axis, and revolves once in 23 hours 56 minutes 4 seconds. What is its velocity in miles per hour? *Ans.*, 787·5.

(3.) A point is in latitude 58°. Take the earth to be a sphere of 4,000 miles radius, find the linear velocity of the point due to the earth's rotation?

Ans., 556·4 miles per hour.

(4.) The average radius of the rim of a fly-wheel is 10 feet. When the wheel makes 150 revolutions per minute what is the average velocity of the rim?

Ans., 157·1 feet per second.

(5.) An acceleration of 1 turn per minute every second; how much is this in radians per second per second?

Ans., ·1047.

(6.) A wheel is revolving at the rate of 90 turns a minute. What is its angular velocity in radians per second?

A point on the wheel is 6 feet from the axis; what is its linear speed? If its distance from the centre be increased by 50 per cent., what does its speed become? If at the same time the speed of the wheel increases 50 per cent., what is now the linear speed of the point?

Ans., 9·425 radians per second; 56·55 feet per second; 84·82 feet per second; 127·2 feet per second.

(7.) There is a lever, AO, 30 inches long, which works about an axis at O. The lever is made to turn by applying a force at a point B in AO, 15 inches from O, so that B receives a velocity of 2 feet per second. What is the angular velocity of the lever?

If the same velocity had been given to the point A instead of B, what would the angular velocity have been?

Ans., 1·6 radians per second; 0·8 radian per second.

114. If a line, AB, makes an angle θ with the horizontal the **projection of its length** on the horizontal is $AB \cos \theta$

Its projection on a vertical line is AB sin θ.

If a plane area of A square inches is inclined at an angle θ with the horizontal, its area as **projected on the horizontal** is A cos θ square inches.

Try to prove that this must be so by dividing the area into strips by horizontal lines.

Exercises.—(1.) A plane area of 35 square feet is inclined at 20° to the horizontal; find its horizontal and vertical projections.

Ans., 32·89 square feet; 11·97 square feet.

(2.) The cross-section (a cross-section always means a section by a plane at right angles to the axis or line of centres of sections) of a cylinder is a circle of 0·7 inch radius. Find the areas of sections which make angles of 25° and 45° with the cross-section. Note that the cross-section is a projection of any other section.

Ans., 1·699, 2·177 square inches.

(3.) The above cylinder is a tie bar of wrought iron. The total tensile load is 12,000 lbs. How much is this per square inch of the cross-section? How much is it per square inch of either of the other sections?

Ans., 7,794 lbs., 7,063 lbs., 5,512 lbs.

(4.) The cross-section of a pipe is a circle of 15 inches diameter: what is the area in square feet? If 13 gallons flow per second, what is the velocity V_0? What is the area of a section at 28° to the cross-section? What is the velocity V normal to this section, if normal velocity × area = cubic feet per second? Show that V is the resolved part of V_0 in a direction normal to the section.

Ans., 1·228, 1·7 ft. per second; 1·39, 1·5 ft. per second.

(5.) Part of a roof, shown in plan as 4,000 square feet, is inclined at 24° to the horizontal. What is its area?

Ans., 4378·7 square feet.

(6.) A tie bar or short strut of 2 square inches cross-section; what is the area of a section making 45° with the cross-section? If the total tensile or compressive load is 20,000 lbs., how much is this per square inch on each of the sections? Resolve the total load normal to and tangential to the oblique section, and find how much it is per square inch each way.

Ans., 2·828 square inches; 10,000 lbs., 7,070 lbs., 5,000 lbs.

115. Vertical Line.—A line showing the direction in which that force which we call the resultant force of gravity acts. It is a line at right angles to the surface of still water or mercury

116. Level Surface.—A surface like that of a still lake everywhere at the same level, and everywhere at right angles to the force of gravity or other volumetric force which is acting upon matter. It is not a plane surface.

117. Curvature.—For any curve we can find at any place what circle will best coincide with the curve just there. The radius of this circle is called the *radius of curvature* at the place. But since we say, for instance, that a railway line curves much, when we mean that the radius is small, the name *curvature* is always given to the reciprocal of the radius. Thus if the radius is 8 feet, we say that the curvature is $\frac{1}{8}$. If at another place the curvature is $\frac{1}{9}$, the change of curvature in going from the one place to the other is the difference between these two fractions.

Example. In making 100 steps round a curve, my compass showing the direction of motion changes from N. to N.E. What is the average curvature?—Answer: From N. to N.E. is 45 degrees or 0·7854 radians, and this divided by 100 steps or ·007854 radians per step, is the average curvature. The reciprocal of this, or 127·3 steps, is the radius of curvature, if the curvature is constant—that is, if the curve is an arc of a circle.

Many UNPRACTICAL RULES will be found in books requiring us to draw a tangent to a given curve at a given point, or to find its curvature there by trial. These are only academic suggestions. If half a dozen students get tracings of the same curve, with two specified points, to measure the angle between the tangents there, they will obtain six very different answers.

118. *Exercise.*—Through what angle must a rail 10 feet long be bent to fit a curve of half a mile radius?

Ans., 0·22 degrees.

119. The System of Teaching Mathematics which I have brought before you is my own, and in part it has been well tried already; it is the outcome of a lifetime of teaching men and boys, of experience of the ways of thinking of all kinds of men and boys in all kinds of schools, night classes, and colleges; in factories and offices; in both scientific and in professional societies. My method is suited to the ways of thinking of average students. The orthodox method is suited to the ways of thinking of old philosophers only. I may say, however, that although I call this system my own, I know that it must be the ideal

system of all teachers who think for themselves. For example, I find it almost exactly sketched by Mr. Oliver Heaviside in his book "Electromagnetic Theory," Vol. I., page 148:—

"... It is worth while to point out that by means of the addition property of vectors a good deal of geometry can be simply done—better than by Euclid, a considerable part of whose 12 books consists of examples of how not to do it. There is a Society for the Improvement of Geometrical Teaching. I have no knowledge of its work; but as to the need of improvement there can be no question while the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then, not Euclid, but algebra. Next, not Euclid, but practical geometry, solid as well as plane; not demonstrations, but to make acquaintance. Then, not Euclid, but elementary vectors, conjoined with algebra and applied to geometry. Addition first, then the scalar product.

"This covers a large ground. When more advanced bring in the vector product. Elementary calculus should go on simultaneously, and come into the vector algebraic geometry after a bit. Euclid might be an extra course for learned men like Homer. But Euclid for children is barbarous."

My treatment of vectors is modelled on the most elementary part of Mr. Heaviside's book.

120. Geometry ought to be practised not earlier than algebra. As soon as a student knows how to evaluate formulæ he may go through some such course as the following, illustrating his study by practical geometry; and assuming the truth of certain things from his own observation and his teacher's statements, he ought to prove the truth of other things which may not be so obviously true. There is a danger that a teacher will give a long useless course on practical geometry or a long course of abstract reasoning. What is wanted is a common-sense treatment of the subject such as will interest the average student, using the kind of reasoning that every thinking man employs in matters in which he is interested. Here is a sort of syllabus:—

121. The usual definitions. The measurement of angles in degrees, in radians, in right angles, and in revolutions. Draw any triangle, measure the angles, and add them together. Test the accuracy of your graphical work. Bisect any angle and any line; draw perpendiculars to lines. Construct triangles when given the three sides or two sides and the contained angle, or one side and the adjacent angles. Learn how to draw parallel lines. Get acquainted with the fact that a straight line drawn parallel to the base of a triangle divides the sides into proportionate segments. Prove that the bisector of an angle of a triangle divides

the opposite side into segments proportional to the other sides. In equiangular triangles the sides are in the same proportions. Divide a straight line into parts proportional to those of a given divided line. The graphical methods of quickly dividing lines into many parts in certain proportions, by means of strips of squared tracing paper, and other labour-saving drawing office methods of working ought to be given to students, not as things to be well remembered, but merely as illustrative exercises, for the wise student will load his memory only with a few important principles. It is most important that early in this work the student should draw a right angled triangle, measure the sides, calculate the sine, cosine, tangent, &c., of the angles to compare with what the tables give, and he will have no difficulty in working simple problems on heights and distances, if he is not frightened by the information that this is a new and very advanced science called Trigonometry.

There are only a few important principles which ought to be proved and illustrated and dwelt upon.

122. Define similar plane and other figures. Similar areas are in proportion to the squares of their like dimensions. Similar volumes are in proportion to the cubes of their like dimensions.

Prove that the area of a parallelogram is the length of a side multiplied by the perpendicular distance between this and the opposite side; that the area of a triangle is half the product of a side and the perpendicular on it from the opposite angle. Try if this gives the same answer as half the product of the lengths of two sides and the sine of the included angle.

Show that in a parallelogram the complements of the parallelograms about the diagonal are equal. Prove the relations between the squares of the lengths of the sides of a right angled triangle, and show that

$$\sin^2 A + \cos^2 A = 1.$$

I am afraid that all this is frightfully unorthodox. The idea of replacing geometrical philosophy by arithmetical juggling is scorned by the modern mathematician. But who knows whether Euclid himself would not have used my method if he had not been so hopelessly ignorant of arithmetic.

Show that the first ten propositions of the 2nd book of Euclid are identical with certain elementary algebraic statements, and the 14th proposition is an extraction of

a square root. Divide a number n into two parts x and $n - x$, so that $n(n - x) = x^2$. This is the 11th proposition. Given two sides of a triangle and the contained angle, show how we find (1) the third side, (2) the other angles.

The solution of triangles ought to be made as easy as possible, and there is no difficulty about this if teachers will only reflect on the great amount of unessential matter that is usually put before students. Hardly anybody now-a-days needs any part of the unessential (except for examination purposes) excrescences which have grown round the subject of the solution of triangles. They were never used except by about three surveyors, and even the surveyor who gets paid for making surveys nowadays almost never makes a survey, because he has his ordnance map.

123. I have spoken of the **first and sixth books of Euclid,** so easy to make interesting to the average student, so stupifying as usually given. The **fifth** book is equivalent to two or three lines of the most elementary algebra, such as, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$, &c. I would impress upon a student the most general of all such propositions; that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{ma + nb}{pa + qb} = \frac{mc + nd}{pc + qd}$, whatever $m, n, p,$ and q may be.

124. As for the **third** book of Euclid, I would simply illustrate propositions 1–19 by drawing, and assume the proofs to be unnecessary. On some of the following propositions there might be a little abstract reasoning, but not until the student has illustrated their truth by actual measurement.

Prove that the angle at the centre is double the angle at the circumference when they have the same part of the circumference for their base, and that angles in the same segment are equal. The angle in a semicircle is a right angle. Also that the sum of the opposite angles of a quadrilateral inscribed in a circle is 180°. Also, the angle between a tangent and a chord is equal to the angle in one of the segments of the circle cut off by the chord. Prove that the products of the parts of chords of a circle cutting one another internally or externally are equal, and consider the special case of one of these being a tangent.

125 In the **fourth** book of Euclid only one thing need be attended to—namely, how to

Inscribe a circle in a given triangle.

126. Of the 7th, 8th, 9th, and 10th books of Euclid I need not speak here, as they deal with arithmetic. Euclid did not, it is true, follow the mystical philosophy of numbers of Pythagoras and Plato, but these books are so utterly useless from every point of view that even the most orthodox of pedants allows them to be replaced by elementary arithmetic. There is some hope, therefore, of the rest of Euclid, and of all the recent pretences at reform of Euclid disappearing from our school teaching in time.

Rules in **Mensuration** ought to be stated as formulæ, and proved, if the proofs are easy, as part of the geometrical work. Surely it is an abominable thing to maintain the present artificial distinction between geometry and mensuration when they are both really the same, and to scorn arithmetic, which we now know, because Euclid, being ignorant, did not use it. The centre of area of a parallelogram is at the intersection of the diagonals. The centre of area of a triangle is one-third of the way along the bisector of a side towards the opposite angle. The circumference of a circle is $3{\cdot}14159d$ or πd. The area of a circle is $\frac{\pi}{4}d^2$ or πr^2. The area of an irregular polygon is found by dividing it into triangles whose areas may be added up. The area of a trapezium is half the sum of the parallel sides multiplied by the perpendicular distance between them. The area of the segment of a circle:—Find the area of the sector having the same arc (or $\frac{1}{2}$ arc $\times$ radius, or $\frac{1}{2}r\theta^2$ if θ (in radians) is the central angle subtended), and find the area of the triangle formed by the chord and the two radii, and subtract.

There are some interesting rules which give the answer with a close approximation to accuracy, such as:—

Area of segment smaller than a semicircle.

If h is the height and c is the chord,

$$\text{Area} = \frac{h^3}{2c} + \tfrac{2}{3}c\,h.$$

Approximate rule for s, the length of a circular arc. $s = \frac{1}{3}(8l - \text{L})$, where l is the chord of half the arc, and L is the chord of the whole arc.

We ought not to compel a student to keep in his memory more than a very few of the rules of mensuration. The most important thing is for a student, when he gets a problem to work, to be able to refer at once to some book in which the rules are clearly stated. In practical work a man may refer to any book for assistance; if I had my way, every candidate

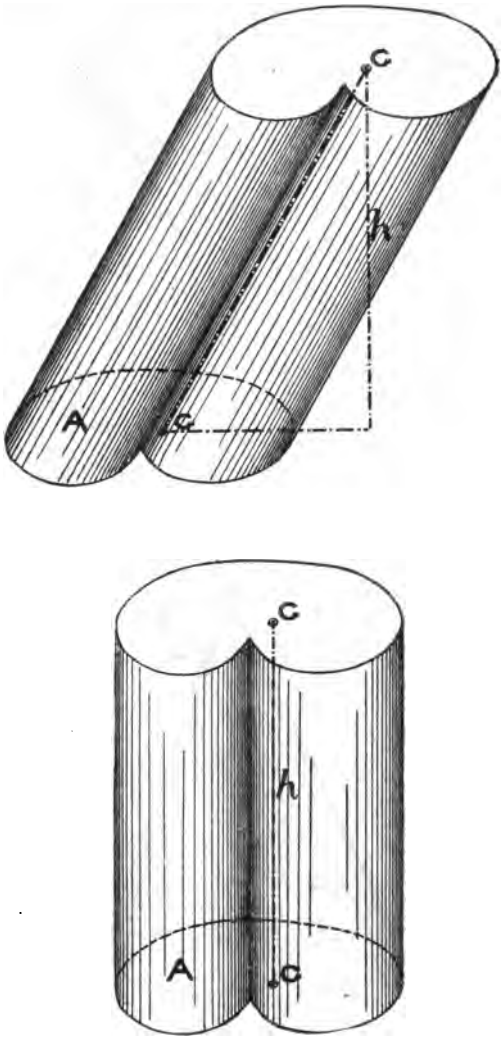


FIG. 48.

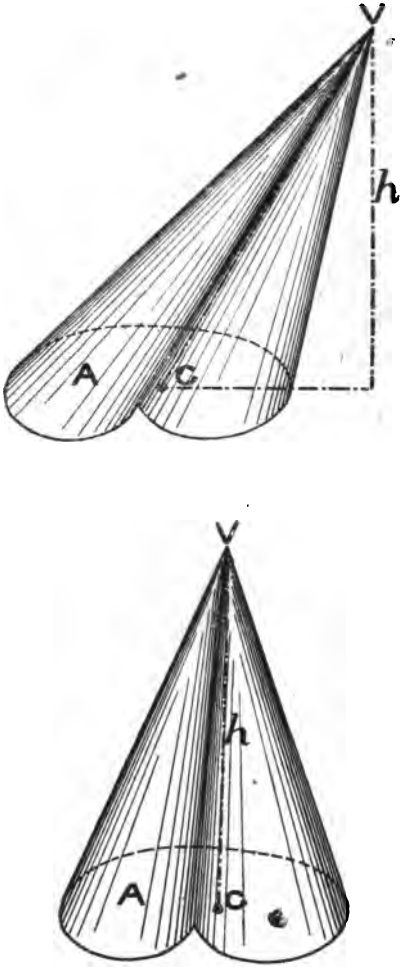


FIG. 49

at an examination would be allowed to bring into the examination room his favourite text-books.

The 11th and 12th books of Euclid are now replaced by **Descriptive Geometry** and Mensuration.

Projections of points and lines. Traces of lines and planes Given two or three planes, by their traces find their intersections and the angles they make with one another. Find a plane containing a given line and point, or containing three given points. Find where a given line and plane meet and the angle between them. Find the angle between two given lines, &c., &c. The average student soon finds it easy to work simple exercises on cones and cylinders and their intersections. The reasons for the following mensuration rules are easily understood.

Prism or Cylinder.—Fig. 48 shows portions ot prisms bounded by parallel ends. The cubic content is always the area of the base multiplied by the perpendicular height, or what comes to the same thing, the area of cross section by length along the axis. The axis of a prism goes through the centres of gravity of the ends. The centre of gravity of a prismatic body is half way along the axis.

Fig. 49 shows portions of **pyramids or cones** bounded by a plane base. The cubic content is found by multiplying the area of the base by one-third of the *perpendicular* height from base to vertex. The centre of gravity of a right cone is one quarter of the way along the axis from the base. The axis of such a body joins the centre of area of the base and the vertex. The curved area is easily constructed by practical geometry. The curved area of a right circular cone is half the circumference of the base multiplied by the slant side.

The volume of a ring is equal to the circumference of the circle which passes through the centres of area of the cross-sections, multiplied by the area of the cross-section. Thus if R is the radius of the central circle and r is the radius of a circular section of a ring, the volume is $V = 2 \pi R \times \pi r^2$ or $2 \pi^2 R r^2$ - - - (1).

The area of a ring is the perimeter of a cross-section multiplied by the circumference of the circle which passes through the centres of all the perimeters. Thus, in the above ring of circular section the area A is

$$A = 4 \pi^2 R r \text{ - - - (2).}$$

Surface of a sphere. Multiply the diameter by the circumference or take four times the area of a diametral circle, or $S = \pi d^2$. Curved surface of a right circular cylinder;

multiply the circumference by the length or $S = 2 \pi r L$. The curved surface of a spherical segment is equal to the curved surface of a cylinder of the same height as the segment, its base being a great circle of the sphere. Curved surface of a right circular cone; multiply half the slant height l by the circumference of the base where radius is r, so that $S = \pi l r$.

Area of an ellipse. Multiply the product of the major and minor axes by $\pi/4$. If the semi-axes are a and b, $A = \pi a b$.

Volume of a sphere. $\frac{\pi d^3}{6}$ or, $\frac{4 \pi r^3}{3}$.

Volume of a plate; area of plate multiplied by its thickness.

Volume of the segment of a sphere; subtract twice the height of the segment from three times the diameter of the sphere, multiply the remainder by the square of the height and by $\pi/6$ or ·5236.

The volume and surface of a sphere are each two-thirds of that of the cylinder which just encloses it.

127. The student ought to notice the importance in every case of writing out **his rule as a formula.**

For example.—A hollow circular **cylinder** of external radius R and internal r and length l, has a volume V, a curved surface S, and a weight W (if w is the weight per unit volume:—

$$V = \pi (R^2 - r^2) l,$$
$$S = 2 \pi (R + r) l,$$
$$W = \pi w (R^2 - r^2) l.$$

Now not only may we be asked to calculate V or S or W, but we may be given V or S or W to calculate some of the dimensions.

Exercises.—(1.) If $R = 2$, $r = 1$, $l = 5$, find V and S.
Ans., V = 47·12.
S = 94·25.

(2.) If V = 160 cubic inches, $l = 7$ inches, $r = 2$ inches, find R. *Ans.*, R = 3·358.

(3.) If W = 20 lbs., R = 4 inches, $r = 2{\cdot}5$ inches, $w =$ 0·3 lb. per cubic inch, find l. *Ans.*, $l = 2{\cdot}177$.

(4.) If V = 230 cubic inches, S = 110 square inches' find $R - r$.

Dividing the formula for V by the formula for S we get

$$\frac{230}{110} = \tfrac{1}{2}(R - r) \text{ so that } R - r = 4{\cdot}182.$$

128. Or taking the formula for **a ring** given above,

$$V = 2\pi^2 R r^2,\ A = 4\pi^2 R r.$$

Exercises.—(1.) If R = 5, r = 1, find V and A.

Ans., V = 98·68, A = 197·4.

(2.) If A = 230 cubic inches, V = 190 square inches find r.

Ans., $\frac{V}{A} = \frac{1}{2} r$, so that $r = 2 \times \frac{190}{230}$ or 1·652 inches.

R is now found to be 3·526.

(3.) If V = 120 cubic inches and R = 10 inches, find r.

Ans., r = ·7798.

129. *Exercises.* (1). Compare with No. 16, Art. 130. A spherical shell, outside diameter 12 inches, weighs 100 lb. The material is such that 1 cubic inch weighs ·26 lb. Find the internal diameter which I here call x.

$$(12^3 \times {\cdot}5236 - x^3 \times {\cdot}5236) \times {\cdot}26 = 100$$

so that it is easy to find x = 9·977.

(2). Compare with No. 10, Art. 130.

A hollow cylinder of brass (0·3 lb. per cubic inch) is 11 inches long and 4 inches internal diameter, its weight is 40 lb.; find its external diameter, which I here call x.

$$(x^2 \times {\cdot}7854 - 4^2 \times {\cdot}7854)\, 11 \times 0{\cdot}3 = 40;$$

so that it is easy to find x = 5·607.

(3). A cast iron wheel (0·26 lb. per cubic inch) weighs 20,000 lb.; the rim has a rectangular section, thickness radially x, size the other way 1·5 x. The inside radius of the rim is 10 x. Find the actual sizes. Evidently

$$(x \times 1{\cdot}5x \times 2\pi \times 10{\cdot}5x)\, 0{\cdot}26 = 20{,}000.$$

Hence x = 9·194 inches

130. *Exercises.* (1). Find the area and circumference of a circle of 3·2 inches diameter.

Ans., 8·041 square inches; 10·05 inches.

(2). Find the area of a triangle whose sides are 2 inches and 3·4 inches, the angle between being 74°.

Ans., 3·268 square inches

(3.) The ordinates of a curve 1 inch apart are 4·60, 4·70, 4·92, 5·14, 4·84, 4·42, 4·20. Find the area between the first and last by Simpson's rule. Test your answer by drawing, using a planimeter, also by taking the average of the six mid-ordinates. *Ans.*, 28·45 square inches.

(4.) Segment of a circle, chord 30 inches, height 4½ inches Find the area. *Ans.*, 91·35 square inches.

If this were a parabolic segment its area would be two-thirds of the chord multiplied by the height, or 90 square inches.

(5.) Find the area of the sector of a circle and the length of its arc, radius 6 inches, angle 50°.

Ans., 15·71 square inches; 5·236 inches.

(6.) Find the area and volume of a sphere of 10 inches radius. *Ans.*, 1256·6 square inches; 4180 cubic inches.

(7.) Segment of sphere, height 6 inches, diameter of base 22·5 inches. Find volume. *Ans.*, 1046 cubic inches.

(8.) A right circular cone, radius of base 3·42 inches, height 6·42 inches; find the area of its curved surface; also find its volume.

Ans., 78·14 square inches; 78·63 cubic inches.

(9.) An anchor ring has a mean radius of 4·2 inches, and the radius of a transverse circular section is 1·05 inch; find the area of its surface, and also its volume.

Ans., 174·1 square inches; 91·39 cubic inches.

(10.) A hollow cylinder is 8·5 inches long; its external and internal diameters are 5 inches and 3·25 inches; find its volume and the area of its curved surface.

Ans., 96·38 cubic inches; 220·4 square inches.

(11.) A piece of round copper wire 100 feet long weighs 4·3 lb.; find its diameter. A cubic inch of copper weighs 0·32 lb. *Ans.*, 0·119 inch.

(12.) Find the area of the convex surface of a conical frustum 7·2 feet high, the radii of whose ends are 4·24 feet and 5·16 feet; also find the volume.

Ans., 214·3 square inches; 501·3 cubic inches.

(13.) Find the volume and weight of the rim of a cast-iron wheel of square section, the inside and outside diameters being 15 feet and 13 feet 6 inches.

Ans., 25·19 cubic feet; 5·05 tons.

(14.) The diameter of a cylinder is 4 feet; find the area of the elliptic section made by a plane inclined to the axis of the cylinder at 50°. *Ans.*, 16·36 square feet.